

Lifting proof theory to the countable ordinals II: second-order indescribable cardinals

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Abstract

We show that the existence of a Π_N^1 -indescribable cardinal over the Zermelo-Fraenkel's set theory \mathbf{ZF} is proof-theoretically reducible to iterations of Mostowski collapsings and lower Mahlo operations. Furthermore we describe a proof-theoretic bound on definable countable ordinals whose existence is provable from the existence of second order indescribable cardinals over \mathbf{ZF} .

1 Introduction

In [3] we showed that the existence of a weakly compact, i.e., Π_1^1 -indescribable cardinal over the Zermelo-Fraenkel's set theory \mathbf{ZF} is proof-theoretically reducible to iterations of Mostowski collapsings and Mahlo operations, while in [4] we describe a proof-theoretic bound on definable countable ordinals in \mathbf{ZF} . In this paper we do the same reductions for the existence of a Π_N^1 -indescribable cardinal, cf. Theorems 5.3 and 5.4 in the last section 5.

Let ORD denote the class of all ordinals, $A \subset ORD$ and α a limit ordinal. A is said to be Π_n^1 -*indescribable* in α iff for any Π_n^1 -formula $\varphi(X)$ and any $B \subset ORD$, if $\langle L_\alpha, \in, B \cap \alpha \rangle \models \varphi(B \cap \alpha)$, then there exists a $\beta \in A \cap \alpha$ such that $\langle L_\beta, \in, B \cap \beta \rangle \models \varphi(B \cap \beta)$. Let us write

$$\alpha \in M_n(A) :\Leftrightarrow A \text{ is } \Pi_n^1\text{-indescribable in } \alpha.$$

Also α is said to be Π_n^1 -*indescribable* iff α is Π_n^1 -indescribable in α .

It is not hard to extend the reduction in [3] to Π_n^1 -indescribability. Namely over $\mathbf{ZF} + (V = L)$ the existence of a Π_n^1 -indescribable cardinal is shown to be proof-theoretically reducible to iterations of Mostowski collapsings and the operation M_{n-1} , since a similar reduction has been done for first-order reflecting ordinals in [5].

In this paper we aim a proof-theoretic reduction of Π_n^1 -indescribability in terms of iterations of Mostowski collapsings and the operations M_i for $i < n$.

Though such a reduction was done for recursive analogues, i.e., Π_n -reflecting ordinals and recursively Mahlo operations in [1, 2, 7], our approach is simpler.

First in [3] we rely on a result by R. Jensen [10], which is the case $n = 1$ in a recent result (Theorem 1.2 below) due to J. Bagaia, M. Magidor and H. Sakai [8].

Definition 1.1 Let $A \subset ORD$, and α a limit ordinal.

1. A is said to be *0-stationary* in α iff $\sup(A \cap \alpha) = \alpha$.
2. For $n > 0$, A is said to be *n-stationary* in α iff for every $m < n$ and every $S \subset ORD$, if S is *m-stationary* in α , then there exists a $\beta \in A \cap \alpha$ such that S is *m-stationary* in β .
3. α is said to be *n-stationary* iff $\alpha = \{\beta \in ORD : \beta < \alpha\}$ or ORD is *n-stationary* in α .

Note that A is 1-stationary in α of uncountable cofinality iff $A \cap \alpha$ is stationary in α , i.e., A meets every club subset of α .

Theorem 1.2 (J. Bagaia, M. Magidor and H. Sakai [8]) *Let κ be a regular uncountable cardinal, and $A \subset ORD$. For each $n > 0$, A is $(n + 1)$ -stationary in κ iff A is Π_n^1 -indescribable in κ , over $ZF + (V = L)$.*

Although the theorem is suggestive, we don't rely on it in this paper.

Second our classes $Mh_{k,n}(\vec{\alpha})[\Theta]$ defined in Definition 2.3 to resolve or approximate Π_N^1 -indescribability are defined from finite *sequences* of ordinals $\vec{\alpha}$. In [1, 2, 7], our ramification process is akin to a tower, i.e., has an exponential structure. Here we simplify the complicated process in terms of sequences. Also cf. an ordinal analysis for first-order reflection using reflection configurations by Pohlers and Stegert [11].

Let us mention the contents of this paper.

In the next section 2 iterated Skolem hulls $\mathcal{H}_{\alpha,n}(X)$ of sets X of ordinals, ordinals $\Psi_{\kappa,n}\gamma$ for regular ordinals κ ($\mathcal{K} < \kappa \leq I$), and classes $Mh_{k,n}(\vec{\alpha})[\Theta]$ are defined for finite sequences $\vec{\alpha}$ of ordinals and finite sets Θ of ordinals. It is shown that for each $k < N$ and each $n, m < \omega$, (\mathcal{K} is a Π_N^1 -indescribable cardinal) $\rightarrow \mathcal{K} \in Mh_{k,n}((\omega_m(I+1), \dots, \omega_m(I+1)))[\emptyset]$ in $ZF + (V = L)$. In the third section 3 we introduce a theory for Π_N^1 -indescribable cardinals, which are equivalent to $ZF + (V = L) + (\mathcal{K} \text{ is a } \Pi_N^1\text{-indescribable cardinal})$.

In the section 4 cut inferences are eliminated from operator controlled derivations of Π_k^1 -sentences $\varphi^{V_{\mathcal{K}}}$ over \mathcal{K} . In the last section 5 Theorems 5.3 and 5.4 are concluded.

2 Ordinals for Π_N^1 -indescribable cardinals

In this section iterated Skolem hulls $\mathcal{H}_{\alpha,n}(X)$ of sets X of ordinals, ordinals $\Psi_{\kappa,n}\gamma$ for regular ordinals κ ($\mathcal{K} < \kappa \leq I$) or $\kappa = \omega_1$, ordinals $\Psi_{\mathcal{K},n}^{\vec{\alpha},\Theta}(\alpha)$ and classes

$Mh_{k,n}(\vec{\alpha})[\Theta]$ are defined for sequences $\vec{\alpha}$ of ordinals and finite sets Θ of ordinals. It is shown that for each $k < N$ and each $n, m < \omega$, $\mathcal{K} \in Mh_{k,n}((\omega_m(I+1), \dots, \omega_m(I+1))[\emptyset])$ in $\mathbf{ZF} + (V = L)$ assuming \mathcal{K} is a Π_N^1 -indescribable cardinal, where $\omega_m(I+1)$ occurs $(N-k)$ times in the class $Mh_{k,n}((\omega_m(I+1), \dots, \omega_m(I+1))[\emptyset])$.

Let $ORD \subset V$ denote the class of ordinals, $ORD^\varepsilon \subset V$ and $<^\varepsilon$ be Δ -predicates such that for any transitive and wellfounded model V of $\mathbf{KP}\omega$, $<^\varepsilon$ is a well ordering of type ε_{I+1} on ORD^ε for the order type I of the class ORD in V .

$<^\varepsilon$ is assumed to be a canonical ordering such that $\mathbf{KP}\omega$ proves the fact that $<^\varepsilon$ is a linear ordering, and for any formula φ and each $n < \omega$,

$$\mathbf{KP}\omega \vdash \forall x (\forall y (<^\varepsilon x \varphi(y) \rightarrow \varphi(x)) \rightarrow \forall x <^\varepsilon [\omega_n(I+1)] \varphi(x) \quad (1)$$

for the code $[\omega_n(I+1)] \in ORD^\varepsilon$ of the ‘ordinal’ $\omega_n(I+1)$.

For a definition of Δ -predicates ORD^ε and $<^\varepsilon$, and a proof of (1), cf. [4].

In the definition of ORD^ε and $<^\varepsilon$, I with its code $[I] = \langle 1, 0 \rangle$ is *intended* to denote the least weakly inaccessible cardinal above the least Π_N^1 -indescribable cardinal \mathcal{K} , though we *do not assume* the existence of weakly inaccessible cardinals above \mathcal{K} anywhere in this paper. We are working in $\mathbf{ZF} + (V = L)$ assuming \mathcal{K} is a regular cardinal.

Let

$$Reg := \{\omega_1\} \cup \{\kappa < I : \mathcal{K} < \kappa \text{ is regular}\}$$

while $Reg^+ := Reg \cup \{I\}$. $\kappa, \lambda, \rho, \pi$ denote elements of Reg . κ^+ denotes the least regular ordinal above κ . Θ denotes finite sets of ordinals $\leq \mathcal{K}$. $\Theta \subset_{fin} X$ iff Θ is a finite subset of X . ORD denotes the class of ordinals less than I , while ORD^ε the class of codes of ordinals less than the next epsilon number ε_{I+1} to I .

For admissible ordinals σ and $X \subset L_\sigma$, $\text{Hull}_{\Sigma_n}^\sigma(X)$ denotes the Σ_n -Skolem hull of X over L_σ , cf. [4]. $F(y) = F^{\Sigma_n}(y; \sigma, X)$ denotes the Mostowski collapsing $F : \text{Hull}_{\Sigma_n}^\sigma(X) \leftrightarrow L_\gamma$ of $\text{Hull}_{\Sigma_n}^\sigma(X)$ for a γ . Let $F^{\Sigma_n}(\sigma; \sigma, X) := \gamma$. When $\sigma = I$, we write $F_X^{\Sigma_n}(y)$ for $F^{\Sigma_n}(y; I, X)$.

In what follows up to the last section 5, $n \geq 1$ denotes a fixed positive integer.

$Code^\varepsilon$ denotes the union of L_I and the codes ORD^ε of ordinals $< \varepsilon_{I+1}$. On $Code^\varepsilon$, $[x] \in^\varepsilon [y] :\Leftrightarrow x \in y$. For simplicity let us identify the code $x \in Code^\varepsilon$ with the ‘set’ coded by x , and $\in^\varepsilon [x]$ is denoted by $\in [x]$, resp. when no confusion likely occurs.

Let $\alpha < \omega_{n+1}(I+1)$ be ordinals, $X \subset L_I$ sets, $n < \omega$, $0 \leq k \leq N$, and $\kappa \in Reg^+$ uncountable regular cardinals $\leq I$.

Define simultaneously classes $\mathcal{H}_{\alpha,n}(X) \subset L_I \cup \{x \in ORD^\varepsilon : x < \omega_n(I+1)\}$, and ordinals $\Psi_{\kappa,n}\alpha$ as in [4]. We see that $\mathcal{H}_{\alpha,n}(X)$ and $\Psi_{\kappa,n}\alpha$ are (first-order) definable as a fixed point in $\mathbf{ZF} + (V = L)$, cf. Proposition 2.4.

Definition 2.1 $\mathcal{H}_{\alpha,n}(X)$ is the Skolem hull of $\{0, \mathcal{K}, I\} \cup X$ under the functions $+, \alpha \mapsto \omega^\alpha, \kappa \mapsto \kappa^+ (\mathcal{K} \leq \kappa \in Reg), (\kappa, \gamma) \mapsto \Psi_{\kappa,n}\gamma (\gamma < \alpha, \kappa \in Reg^+)$, the

Skolem hullings:

$$X \mapsto \text{Hull}_{\Sigma_n}^I(X \cap I)$$

and the Mostowski collapsing functions

$$x = \Psi_{\kappa,n}\gamma \mapsto F_{x \cup \{\kappa\}}^{\Sigma_1}(\kappa \in \text{Reg})$$

and

$$x = \Psi_{I,n}\gamma \mapsto F_x^{\Sigma_n}$$

1. (Inductive definition of $\mathcal{H}_{\alpha,n}(X)$).

- (a) $\{0, \omega_1, \mathcal{K}, I\} \cup X \subset \mathcal{H}_{\alpha,n}(X)$.
- (b) $x, y \in \mathcal{H}_{\alpha,n}(X) \Rightarrow x + y \in \mathcal{H}_{\alpha,n}(X)$, and $x \in \mathcal{H}_{\alpha,n}(X) \cap \omega_n(I + 1) \Rightarrow \omega^x \in \mathcal{H}_{\alpha,n}(X)$.
- (c) $\mathcal{K} \leq \kappa \in \mathcal{H}_{\alpha,n}(X) \cap \text{Reg} \Rightarrow \kappa^+ \in \mathcal{H}_{\alpha,n}(X)$.
- (d) $\gamma \in \mathcal{H}_{\alpha,n}(X) \cap \alpha \Rightarrow \Psi_{I,n}\gamma \in \mathcal{H}_{\alpha,n}(X)$.
- (e) If $\kappa \in \mathcal{H}_{\alpha,n}(X) \cap \text{Reg}$ and $\gamma \in \mathcal{H}_{\alpha,n}(X) \cap \alpha$ then $\Psi_{\kappa,n}\gamma \in \mathcal{H}_{\alpha,n}(X)$.
- (f)

$$\text{Hull}_{\Sigma_n}^I(\mathcal{H}_{\alpha,n}(X) \cap L_I) \cap \text{Code}^\varepsilon \subset \mathcal{H}_{\alpha,n}(X).$$

Namely for any Σ_n -formula $\varphi[x, \vec{y}]$ in the language $\{\in, =\}$ and parameters $\vec{a} \subset \mathcal{H}_{\alpha,n}(X) \cap L_I$, if $b \in L_I$, $(L_I, \in) \models \varphi[b, \vec{a}]$ and $(L_I, \in) \models \exists! x \varphi[x, \vec{a}]$, then $b \in \mathcal{H}_{\alpha,n}(X)$.

- (g) If $\kappa \in \mathcal{H}_{\alpha,n}(X) \cap \text{Reg}$, $\gamma \in \mathcal{H}_{\alpha,n}(X) \cap \alpha$, $x = \Psi_{\kappa,n}\gamma \in \mathcal{H}_{\alpha,n}(X)$, $\kappa \in \mathcal{H}_{\gamma,n}(\kappa)$ and $\delta \in (\text{Hull}_{\Sigma_1}^I(x \cup \{\kappa\}) \cup \{I\}) \cap \mathcal{H}_{\alpha,n}(X)$, then $F_{x \cup \{\kappa\}}^{\Sigma_1}(\delta) \in \mathcal{H}_{\alpha,n}(X)$.
- (h) If $\gamma \in \mathcal{H}_{\alpha,n}(X) \cap \alpha$, $x = \Psi_{I,n}\gamma \in \mathcal{H}_{\alpha,n}(X)$, and $\delta \in (\text{Hull}_{\Sigma_n}^I(x) \cup \{I\}) \cap \mathcal{H}_{\alpha,n}(X)$, then $F_x^{\Sigma_n}(\delta) \in \mathcal{H}_{\alpha,n}(X)$.

2.

$$\mathcal{H}_{\alpha,n}[Y](X) := \mathcal{H}_{\alpha,n}(Y \cup X)$$

for sets $Y \subset L_I$.

3. (Definition of $\Psi_{\kappa,n}\alpha$).

Assume $\kappa \in \text{Reg}^+$. Then

$$\Psi_{\kappa,n}\alpha := \min(\{\kappa\} \cup \{\beta < \kappa : \mathcal{H}_{\alpha,n}[\{\kappa\}](\beta) \cap \kappa \subset \beta\}).$$

Next classes $Mh_{k,n}(\vec{\alpha})[\Theta]$ of regular cardinals and ordinals $\Psi_{\mathcal{K},n}^{\vec{\alpha},\Theta}(\alpha)$ are defined.

Definition 2.2 1. Let $\vec{\alpha} = (\alpha_0, \dots, \alpha_{m-1})$ be a sequence of ordinals.

(a) length $lh(\vec{\alpha}) := m$, components $\vec{\alpha}(i) := \alpha_i$ and end segments $\vec{\alpha}[i] := (\alpha_i, \dots, \alpha_{m-1})$ for $i < lh(\vec{\alpha})$.

(b) The set of components

$$K(\vec{\alpha}) := \{\vec{\alpha}(i) : i < lh(\vec{\alpha})\} = \{\alpha_0, \dots, \alpha_{m-1}\}.$$

(c) Sequences consisting of a single element (α) is identified with the ordinal α , and \emptyset denotes the *empty sequence*.

(d) For sequence of ordinals $\vec{\nu}$ of the same length, $lh(\vec{\nu}) = lh(\vec{\alpha})$,

$$\vec{\nu} < \vec{\alpha} :\Leftrightarrow \forall i < lh(\vec{\alpha}) [\vec{\nu}(i) < \vec{\alpha}(i)]$$

(e) For ordinals β ,

$$\vec{\alpha} \leq \beta :\Leftrightarrow \forall \alpha \in K(\vec{\alpha}) (\alpha \leq \beta).$$

(f) For sequence of ordinals $\vec{\nu} = (\nu_0, \dots, \nu_{m-1})$ of the same length, $lh(\vec{\nu}) = lh(\vec{\alpha})$ and $i < lh(\vec{\alpha})$

$$(\vec{\nu} \bullet \vec{\alpha})[i] := (\vec{\nu}(i)) * \vec{\alpha}[i+1] = (\nu_i, \alpha_{i+1}, \dots, \alpha_{m-1}).$$

Note that $lh((\vec{\nu} \bullet \vec{\alpha})[i]) = lh(\vec{\alpha}) - i$.

2. For $A \subset ORD$, limit ordinals α and $i \geq 0$

$$\alpha \in M_i(A) :\Leftrightarrow A \text{ is } \Pi_i^1\text{-indescribable in } \alpha (\Leftrightarrow A \text{ is } (i+1)\text{-stationary in } \alpha).$$

Definition 2.3 1. For sequence of ordinals $\vec{\nu}, \vec{\alpha}$ in the same length, let

$$\vec{\nu} \in \mathcal{H}_{\vec{\nu},n}[\Theta](\pi) \cap \vec{\alpha} :\Leftrightarrow \forall i < lh(\vec{\nu}) [\vec{\nu}(i) \in \mathcal{H}_{\vec{\nu}(i),n}[\Theta](\pi) \cap \vec{\alpha}(i)]$$

2. (Definition of $Mh_{k,n}(\vec{\alpha})[\Theta]$) First let for the empty sequence \emptyset

$$\mathcal{K} \in Mh_{N,n}(\emptyset)[\Theta] :\Leftrightarrow \mathcal{K} \in M_N \Leftrightarrow \mathcal{K} \text{ is } \Pi_N^1\text{-indescribable}.$$

The classes $Mh_{k,n}(\vec{\alpha})[\Theta]$ are defined for $n < \omega$, $0 \leq k < N$, $\Theta \subset_{fin} (\mathcal{K} + 1)$, and sequences of ordinals $\vec{\alpha}$ such that $lh(\vec{\alpha}) = N - k$.

$$\pi \in Mh_{k,n}(\vec{\alpha})[\Theta] :\Leftrightarrow \pi \text{ is a regular cardinal } \leq \mathcal{K} \text{ such that} \quad (2)$$

$$\forall \vec{\nu} \in \mathcal{H}_{\vec{\nu},n}[\Theta \cup \{\pi\}](\pi) \cap \vec{\alpha} [\pi \in M_k(\bigcap_{i < lh(\vec{\alpha})} Mh_{k+i,n}((\vec{\nu} \bullet \vec{\alpha})[i])[\Theta \cup \{\pi\}])]$$

3. We say that the class

$$\{\rho \in \bigcap_{i < lh(\vec{\alpha})} Mh_{k+i,n}((\vec{\nu} \bullet \vec{\alpha})[i])[\Theta \cup \{\pi\}] \cap \pi : \mathcal{H}_{\gamma,n}[\Theta \cup \{\pi\}](\rho) \cap \pi \subset \rho\} \quad (3)$$

is the *resolvent class* for $\pi \in Mh_{k,n}(\vec{\alpha})[\Theta]$ with respect to $\vec{\nu}$ and γ .

4. (Definition of $\Psi_{\mathcal{K},n}^{\vec{\alpha},\Theta}(\alpha)$)

For $\vec{\alpha} \leq \alpha$ with $lh(\vec{\alpha}) = N$, let

$$\Psi_{\mathcal{K},n}^{\vec{\alpha},\Theta}(\alpha) := \min(\{\mathcal{K}\} \cup \{\pi \in \bigcap_{i < N} Mh_{i,n}(\vec{\alpha}[i])[\Theta \cup \{\pi\}] \cap \mathcal{K} : \mathcal{H}_{\alpha,n}[\Theta](\pi) \cap \mathcal{K} \subset \pi\}) \quad (4)$$

5.

$$Mh_{k,n}(\vec{\alpha}) := Mh_{k,n}(\vec{\alpha})[\emptyset].$$

x, y, z, \dots range over sets in L_I , $\alpha, \beta, \gamma, \dots$ range over ORD^ε , $\vec{\alpha}, \vec{\nu}, \dots$ range over finite sequences over ORD^ε . φ, τ denote formulae.

The following Proposition 2.4 is easy to see.

Proposition 2.4 *Each of $x = \mathcal{H}_{\alpha,n}(\beta)$ ($\alpha \in ORD^\varepsilon, \beta <^\varepsilon I$), $\beta = \Psi_{\kappa,n}\alpha$ ($\kappa \in R^+$), $x \in Mh_{k,n}(\vec{\alpha})[\Theta]$ and $\beta = \Psi_{\mathcal{K},n}^{\vec{\alpha},\Theta}(\alpha)$ is a Σ_{n+1} -predicate as fixed points in $ZF + (V = L)$.*

Proof. Let us examine the definability of $x \in Mh_{k,n}(\vec{\alpha})[\Theta]$.

Let $\alpha = \max K(\vec{\alpha})$, and $m = lh(\vec{\alpha})$. Then $\pi \in Mh_{k,n}(\vec{\alpha})[\Theta]$ iff $\pi \leq \mathcal{K}$ is regular and there exist a set $y = \mathcal{H}_{\alpha,n}[\Theta \cup \{\pi\}](\pi)$ and a function $\{x_\nu\}_{\nu \in y}$ such that $\forall \nu \in y [x_\nu = \mathcal{H}_{\nu,n}[\Theta \cup \{\pi\}](\pi)]$ and for any sequence $\vec{\nu} \in {}^m y$ (i.e., $\forall i < m = lh(\vec{\nu}) (\vec{\nu}(i) \in y)$) with $\vec{\nu} < \vec{\alpha}$, if $\forall i < lh(\vec{\nu}) (\vec{\nu}(i) \in x_{\vec{\nu}(i)})$, then for any $\Pi_k^1(\pi)$ -sentence θ true on L_π , there exists a $\sigma < \pi$ such that θ holds in L_σ and $\sigma \in \bigcap_{i < lh(\vec{\alpha})} Mh_{k+i,n}((\vec{\nu} \bullet \vec{\alpha})[i])[\Theta \cup \{\pi\}]$. \square

Proposition 2.5 1. Suppose $lh(\vec{\beta}) = lh(\vec{\alpha})$ and $\forall i < lh(\vec{\alpha}) (\vec{\beta}(i) \leq \vec{\alpha}(i))$. Then $\pi \in Mh_{k,n}(\vec{\alpha})[\Theta] \Rightarrow \pi \in Mh_{k,n}(\vec{\beta})[\Theta]$.

2. $\mathcal{K} \in M_{N-1}(Mh_{N-1,n}(\alpha)[\Theta \cup \{\mathcal{K}\}]) \Rightarrow \mathcal{K} \in Mh_{N-1,n}(\alpha)[\Theta \cup \{\mathcal{K}\}]$.

3. $Mh_{k,n}(\vec{\alpha})[\Theta \cup \{\kappa\}] \subset Mh_{k,n}(\vec{\alpha})[\Theta]$.

4. $\rho \in Mh_{k,n}(\vec{\alpha})[\Theta] \Leftrightarrow \rho \in Mh_{k,n}(\vec{\alpha})[\Theta \cup \{\rho\}]$.

5. $\kappa \in Mh_{k,n}(\vec{\alpha})[\Theta] \Leftrightarrow \kappa \in Mh_{k,n}(\vec{\alpha})[\Theta \cup \{\beta\}]$ for any $\beta < \kappa$.

6. $\Theta_1 \subset_{fin} \pi \ \& \ \pi \in M_k(Mh_{k,n}(\vec{\alpha})[\Theta_0]) \Rightarrow \pi \in M_k(Mh_{k,n}(\vec{\alpha})[\Theta_0 \cup \Theta_1])$.

Proof. 2.5.1. This is seen from Definition 2.3, (2).

2.5.2. Suppose $\mathcal{K} \in M_{N-1}(Mh_{N-1,n}(\alpha)[\Theta \cup \{\mathcal{K}\}])$ and $\beta < \alpha$. Then by Proposition 2.5.1 we have $\mathcal{K} \in M_{N-1}(Mh_{N-1,n}(\beta)[\Theta \cup \{\mathcal{K}\}])$, and hence $\mathcal{K} \in Mh_{N-1,n}(\alpha)[\Theta \cup \{\mathcal{K}\}]$.

2.5.3. $\lambda \in Mh_{k,n}(\vec{\alpha})[\Theta \cup \{\kappa\}] \Rightarrow \lambda \in Mh_{k,n}(\vec{\alpha})[\Theta]$ is seen easily by induction on λ from the monotonicity of the operators M_k , $X \subset Y \Rightarrow M_k(X) \subset M_k(Y)$,

and $\mathcal{H}_{\nu,n}[\Theta \cup \{\lambda\}] \subset \mathcal{H}_{\nu,n}[\Theta \cup \{\kappa, \lambda\}]$.

2.5.4. This is seen from Definition 2.3, (2) and Proposition 2.5.3.

2.5.5. By induction on κ . Let $\beta < \kappa$. We have $\vec{\nu} \in \mathcal{H}_{\vec{\nu},n}[\Theta \cup \{\kappa\}](\kappa)$ iff $\vec{\nu} \in \mathcal{H}_{\vec{\nu},n}[\Theta \cup \{\kappa, \beta\}](\kappa)$ since $\beta < \kappa$. It suffices to show for such a $\vec{\nu} < \vec{\alpha}$, $\kappa \in M_k(\bigcap_{i < lh(\vec{\alpha})} Mh_{k+i,n}((\vec{\nu} \bullet \vec{\alpha})[i])[\Theta \cup \{\kappa\}])$ iff $\kappa \in M_k(\bigcap_{i < lh(\vec{\alpha})} Mh_{k+i,n}((\vec{\nu} \bullet \vec{\alpha})[i])[\Theta \cup \{\kappa, \beta\}])$. By IH for any ρ with $\beta < \rho < \kappa$, $\rho \in Mh_{k+i,n}((\vec{\nu} \bullet \vec{\alpha})[i])[\Theta \cup \{\kappa\}]$ iff $\rho \in Mh_{k+i,n}((\vec{\nu} \bullet \vec{\alpha})[i])[\Theta \cup \{\kappa, \beta\}]$. This shows the lemma since for any $\beta < \kappa$ and any $X \subset ORD$, $\kappa \in M_k(X)$ iff $\kappa \in M_k(X \setminus (\beta + 1))$.

2.5.6. This is seen from Proposition 2.5.5. \square

Let $A_n(\alpha)$ denote the conjunction of $\forall \beta <^\varepsilon I \exists! x [x = \mathcal{H}_{\alpha,n}(\beta)]$, $\forall \kappa \in R^+ \forall x [x = \mathcal{H}_{\alpha,n}(\kappa) \rightarrow \exists! \beta < \kappa (\beta = \Psi_{\kappa,n} \alpha)]$ and $\forall \vec{\alpha} \forall k < N \forall \Theta \subset_{fin} (\mathcal{K} + 1)(\max K(\vec{\alpha}) \leq \alpha \ \& \ lh(\vec{\alpha}) + k = N \rightarrow \exists! x (x = Mh_{k,n}(\vec{\alpha})[\Theta]) \ \& \ \exists! \beta (\beta = \Psi_{\mathcal{K},n}^{\vec{\alpha}, \Theta}(\alpha)))$.

$card(x)$ denotes the cardinality of sets x .

Lemma 2.6 *For each $n, m < \omega$, $ZF + (V = L)$ proves the followings.*

1. $y = \mathcal{H}_{\alpha,n}(x) \rightarrow card(y) \leq \max\{card(x), \aleph_0\}$.
2. $\forall \alpha <^\varepsilon \omega_m(I + 1) A_n(\alpha)$.
3. If \mathcal{K} is Π_N^1 -indescribable and $\Theta \subset_{fin} (\mathcal{K} + 1)$, then $\mathcal{K} \in Mh_{N-1,n}(\omega_m(I + 1))[\Theta] \cap M_{N-1}(Mh_{N-1,n}(\omega_m(I + 1))[\Theta])$ for each $m < \omega$.
4. $\pi \in Mh_{k,n}(\vec{\alpha})[\Theta]$ is a Π_{k+1}^1 -class on L_π uniformly for weakly inaccessible cardinals $\pi \leq \mathcal{K}$. This means that for each k, n there exist a Π_{k+1}^1 -formula $m_{k,n}(R_0, R_1)$ and some $r_i \subset L_\mathcal{K}$ depending on $\vec{\alpha}, \Theta$ such that $\pi \in Mh_{k,n}(\vec{\alpha})[\Theta]$ iff $L_\pi \models m_{k,n}(r_0 \cap L_\pi, r_1 \cap L_\pi)$ for any weakly inaccessible cardinals $\pi \leq \mathcal{K}$.

Proof.

2.6.2. We show that $A_n(\alpha)$ is progressive, i.e., $\forall \alpha <^\varepsilon \omega_m(I + 1) [\forall \gamma <^\varepsilon \alpha A_n(\gamma) \rightarrow A_n(\alpha)]$.

Assume $\forall \gamma <^\varepsilon \alpha A_n(\gamma)$ and $\alpha <^\varepsilon \omega_m(I + 1)$. $\forall \beta <^\varepsilon I \exists! x [x = \mathcal{H}_{\alpha,n}(\beta)]$ follows from IH and the Replacement.

Next assume $\kappa \in Reg^+$. Then $\exists! \beta < \kappa (\beta = \Psi_{\kappa,n} \alpha)$ follows from the regularity of κ .

$\max K(\vec{\alpha}) \leq \alpha \rightarrow \exists! x [x = Mh_{k,n}(\vec{\alpha})[\Theta]]$ is easily seen from IH, Proposition 2.4 and Separation $Mh_{k,n}(\vec{\alpha})[\Theta] \subset (\mathcal{K} + 1)$. $\exists! \beta (\beta = \Psi_{\mathcal{K},n}^{\vec{\alpha}, \Theta}(\alpha))$ follows from this.

2.6.3. Suppose \mathcal{K} is Π_N^1 -indescribable. We show the following $B_n(\alpha)$ is progressive in $\alpha \in ORD^\varepsilon$:

$$\begin{aligned} B_n(\alpha) \quad & :\Leftrightarrow \quad \forall \Theta \subset_{fin} (\mathcal{K} + 1) [\alpha \in \mathcal{H}_{\alpha,n}[\Theta \cup \{\mathcal{K}\}](\mathcal{K}) \\ & \rightarrow \mathcal{K} \in Mh_{N-1,n}(\alpha)[\Theta] \cap M_{N-1}(Mh_{N-1,n}(\alpha)[\Theta])]. \end{aligned}$$

Suppose $\forall \xi <^\varepsilon \alpha B_n(\xi)$, $\Theta \subset_{fin} (\mathcal{K}+1)$, and $\alpha \in \mathcal{H}_{\alpha,n}[\Theta \cup \{\mathcal{K}\}](\mathcal{K})$. We have to show that $Mh_{N-1,n}(\alpha)[\Theta]$ is Π_{N-1}^1 -indescribable in \mathcal{K} . $\mathcal{K} \in Mh_{N-1,n}(\alpha)[\Theta]$ follows from $\mathcal{K} \in M_{N-1}(Mh_{N-1,n}(\alpha)[\Theta])$, cf. Proposition 2.5.2.

Let $\sigma(X, m)$ be a universal Π_{N-1}^1 -formula, and assume that $L_{\mathcal{K}} \models \sigma(C_0, m_0)$ for a subset C_0 of \mathcal{K} and an $m_0 < \omega$.

Since $\forall \pi \leq \mathcal{K} [\text{card}(\mathcal{H}_{\alpha,n}[\Theta \cup \{\pi\}](\pi)) \leq \pi]$, pick an injection $f : \mathcal{H}_{\alpha,n}[\Theta \cup \{\mathcal{K}\}](\mathcal{K}) \rightarrow \mathcal{K}$ so that $f''\mathcal{H}_{\alpha,n}[\Theta \cup \{\pi\}](\pi) \subset \pi$ for any weakly inaccessible $\pi \leq \mathcal{K}$.

Let $R_0 = \{f(\alpha)\}$, $R_1 = C_0$, $R_2 = \{f(\xi) : \xi \in \mathcal{H}_{\xi,n}[\Theta \cup \{\mathcal{K}\}](\mathcal{K}) \cap \alpha\}$, $R_3 = \bigcup \{(Mh_{N-1,n}(\xi)[\Theta \cup \{\pi\}] \cap \mathcal{K}) \times \{f(\pi)\} \times \{f(\xi)\} : \xi \in \mathcal{H}_{\xi,n}[\Theta \cup \{\mathcal{K}\}](\mathcal{K}) \cap \alpha, \pi \leq \mathcal{K}\}$, and $R_4 = \{(f(\beta), f(\gamma)) : \{\beta, \gamma\} \subset \mathcal{H}_{\alpha,n}[\Theta \cup \{\mathcal{K}\}](\mathcal{K}), \beta < \gamma\}$.

By IH we have $\forall \xi \in \mathcal{H}_{\xi,n}[\Theta \cup \{\mathcal{K}\}](\mathcal{K}) \cap \alpha [\mathcal{K} \in M_{N-1}(Mh_{N-1,n}(\xi)[\Theta \cup \{\mathcal{K}\}])]$. Hence $\langle L_{\mathcal{K}}, \in, R_i \rangle_{i \leq 4}$ enjoys a Π_N^1 -sentence saying that \mathcal{K} is weakly inaccessible, $R_0 \neq \emptyset$, $\sigma(R_1, m_0)$ and

$$\varphi : \Leftrightarrow \forall m \in \omega \forall C \subset ORD \forall x, y \exists a [R_2(x) \wedge \theta(R_4, y) \wedge \sigma(C, m) \rightarrow R_3(a, y, x) \wedge L_a \models \sigma(C \cap a, m)]$$

where $\theta(R_4, y)$ is a Σ_1^1 -formula such that for any $\pi \leq \mathcal{K}$

$$L_\pi \models \theta(R_4, y) \Leftrightarrow y = f(\pi).$$

By the Π_N^1 -indescribability of \mathcal{K} , pick a $\pi < \mathcal{K}$ such that $\langle L_\pi, \in, R_i \cap L_\pi \rangle_{i \leq 4}$ enjoys the Π_N^1 -sentence.

We claim that $\pi \in Mh_{N-1,n}(\alpha)[\Theta]$ and $L_\pi \models \sigma(C_0 \cap \pi, m_0)$.

π is weakly inaccessible, $f(\alpha) \in \pi$ and $\sigma(C_0 \cap \pi, m_0)$ hold in L_π . We can assume that $\mathcal{H}_{\alpha,n}[\{\mathcal{K}\}](\pi) \cap \mathcal{K} \subset \pi$ since $\{\pi < \mathcal{K} : \mathcal{H}_{\alpha,n}[\{\mathcal{K}\}](\pi) \cap \mathcal{K} \subset \pi\}$ is club in \mathcal{K} .

It remains to see $\forall \xi \in \mathcal{H}_{\xi,n}[\Theta \cup \{\pi\}](\pi) \cap \alpha [\pi \in M_{N-1}(Mh_{N-1,n}(\xi)[\Theta \cup \{\pi\}])]$. This follows from the fact that φ holds in $\langle L_\pi, \in, R_i \cap L_\pi \rangle_{i \leq 4}$, and $\forall \xi \in \mathcal{H}_{\xi,n}[\Theta \cup \{\pi\}](\pi) \cap \alpha [f(\xi) \in \pi]$ by $f''\mathcal{H}_{\alpha,n}[\Theta \cup \{\pi\}](\pi) \subset \pi$ and $\mathcal{H}_{\xi,n}[\Theta \cup \{\pi\}](\pi) \subset \mathcal{H}_{\xi,n}[\Theta \cup \{\mathcal{K}\}](\mathcal{K})$.

Thus $\mathcal{K} \in M_{N-1}(Mh_{N-1,n}(\alpha)[\Theta])$.

2.6.4. This is seen as in Lemma 2.6.3 using a universal Π_k^1 -formula and an injection $f : \mathcal{H}_{\alpha,n}[\Theta \cup \{\mathcal{K}\}](\mathcal{K}) \rightarrow \mathcal{K}$ for $\alpha = \max K(\vec{\alpha})$ so that $f''\mathcal{H}_{\alpha,n}[\Theta \cup \{\pi\}](\pi) \subset \pi$ for any weakly inaccessible $\pi \leq \mathcal{K}$. \square

Lemma 2.7 *For each $m < \omega$, $\mathbf{ZF} + (V = L)$ proves the following.*

*Let $\pi \in Mh_{k+1,n}(\vec{\alpha})[\Theta]$. Then for any ordinals $\gamma < \omega_m(I+1)$, $\pi \in Mh_{k,n}((\gamma) * \vec{\alpha})[\Theta]$.*

Proof. Suppose $\pi \in Mh_{k+1,n}(\vec{\alpha})[\Theta]$. We show $\pi \in Mh_{k,n}((\gamma) * \vec{\alpha})[\Theta]$ by induction on γ .

Let $X = \bigcap_{i < lh(\vec{\alpha})} Mh_{k+1+i,n}((\vec{\nu} \bullet \vec{\alpha})[i])[\Theta \cup \{\pi\}]$ and $Y = Mh_{k,n}((\delta) * \vec{\alpha})[\Theta \cup \{\pi\}]$ for a sequence $\vec{\nu} \in \mathcal{H}_{\vec{\nu},n}[\Theta \cup \{\pi\}](\pi) \cap \vec{\alpha}$ and an ordinal $\delta \in \mathcal{H}_{\delta,n}[\Theta \cup \{\pi\}](\pi) \cap \gamma$. We show $\pi \in M_k(X \cap Y)$, cf. the definition (2).

The assumption $\pi \in Mh_{k+1,n}(\vec{\alpha})[\Theta]$ yields $\pi \in M_{k+1}(X)$. On the other side IH yields $\pi \in Mh_{k,n}((\delta) * \vec{\alpha})[\Theta]$. By Proposition 2.5.4 we have $\pi \in Y$.

Since $\pi \in Y$ is a Π_{k+1}^1 -sentence on L_π by Lemma 2.6.4, $\pi \in M_{k+1}(X)$ yields $\pi \in M_{k+1}(X \cap Y)$, a fortiori $\pi \in M_k(X \cap Y)$. Thus $\pi \in Mh_{k,n}((\gamma) * \vec{\alpha})[\Theta]$ is shown. \square

Theorem 2.8 *For each $m < \omega$ and $k < N$, $\mathbf{ZF} + (V = L)$ proves the following.*

Let \mathcal{K} be a Π_N^1 -indescribable cardinal.

Then $\mathcal{K} \in Mh_{k,n}(\vec{\beta})[\emptyset]$, $\mathcal{K} \in Mh_{k,n}(\vec{\beta})[\Theta]$ and $\exists \kappa < \mathcal{K}(\kappa = \Psi_{\mathcal{K},n}^{\vec{\alpha},\Theta}(\alpha))$ for any finite set $\Theta \subset_{fin} (\mathcal{K} + 1)$ and any $\vec{\beta}, \alpha, \vec{\alpha}$ with $lh(\vec{\beta}) = N - k$, $K(\vec{\beta}) < \omega_m(I + 1)$ and $\vec{\alpha} \leq \alpha < \omega_m(I + 1)$.

Proof. By Lemma 2.6.4 $\bigcap_{k < N} Mh_{k,n}(\vec{\alpha}[k])[\Theta]$ is a Π_N^1 -class. Hence if a Π_N^1 -indescribable $\mathcal{K} \in \bigcap_{k < N} Mh_{k,n}(\vec{\alpha}[k])[\Theta]$, then $\mathcal{K} \in M_N(\bigcap_{k < N} Mh_{k,n}(\vec{\alpha}[k])[\Theta])$, a fortiori, $\mathcal{K} \in M_0(\bigcap_{k < N} Mh_{k,n}(\vec{\alpha}[k])[\Theta])$. Since $\{\kappa < \mathcal{K} : \mathcal{H}_{\alpha,n}[\Theta](\kappa) \cap \mathcal{K} \subset \kappa\}$ is club in \mathcal{K} , $\mathcal{K} \in M_0(\bigcap_{k < N} Mh_{k,n}(\vec{\alpha}[k])[\Theta])$ yields $\exists \kappa < \mathcal{K}(\kappa = \Psi_{\mathcal{K},n}^{\vec{\alpha},\Theta}(\alpha))$ for $\vec{\alpha} \leq \alpha$.

By metainduction on $N - k$ using Lemmata 2.6.3 and 2.7 we see $\mathcal{K} \in Mh_{k,n}(\vec{\beta})[\emptyset]$ or equivalently $\mathcal{K} \in Mh_{k,n}(\vec{\beta})[\{\mathcal{K}\}]$. Proposition 2.5.6 yields $\mathcal{K} \in Mh_{k,n}(\vec{\beta})[\Theta \cup \{\mathcal{K}\}]$ for any $\Theta < \mathcal{K}$. \square

3 A theory for Π_N^1 -indescribable cardinal

In this section the set theory $\mathbf{ZF} + (V = L) + (\mathcal{K} \text{ is } \Pi_N^1\text{-indescribable})$ is paraphrased to another set theory $\mathbf{T}_N(\mathcal{K}, I)$. Since our formulation of $\mathbf{T}_N(\mathcal{K}, I)$ is the same as in [3, 4], let me define it briefly.

Let \mathcal{K} be the least Π_N^1 -indescribable cardinal for a fixed positive integer $N \geq 1$, and $I > \mathcal{K}$ (be intended to denote) the least weakly inaccessible cardinal above \mathcal{K} . Again note that we do not assume that such an I exists. κ, λ, ρ ranges over uncountable regular ordinals σ such that $\mathcal{K} < \sigma < I$ or $\sigma = \omega_1$.

In the following Definition 3.2, $\text{Hull}_{\Sigma_n}^I(x)$ denotes the Σ_n -Skolem hull of the set x on the universe $L_I = L$, and $\text{Hull}_{\Sigma_n}^I(x) \ni z \mapsto F_x^{\Sigma_n}(z) \in L_y$ its Mostowski collapsing map with $y = F_x^{\Sigma_n}(I)$. Then the predicate P is intended to denote the relation

$$P(\lambda, x, y) \Leftrightarrow x = F_{x \cup \{\lambda\}}^{\Sigma_1}(\lambda) \ \& \ y = F_{x \cup \{\lambda\}}^{\Sigma_1}(I) := \text{rng}(F_{x \cup \{\lambda\}}^{\Sigma_1}) \cap \text{ORD}$$

and the predicate $P_{I,n}(x)$ is intended to denote the relation

$$P_{I,n}(x) \Leftrightarrow x = F_x^{\Sigma_n}(I).$$

Definition 3.1 1. For a formula φ and a set x , φ^x denotes the result of restricting every unbounded quantifier $\exists z, \forall z$ in φ to $\exists z \in x, \forall z \in x$.

2. For natural numbers $k \geq 0$, a Π_k^1 -formula is a formula obtained from a (first-order) formula $\varphi[\vec{X}]$ in the language $\{\in\} \cup \vec{X}$ with unary predicates \vec{X} by applying alternating second-order quantifiers at most k -times, $\forall X_k \exists X_{k-1} \cdots QX_1 \varphi[\vec{X}]$.

Σ_k^1 -formulae are defined dually.

3. Let $\{X_i\}_{i < \omega}$ be the list of second-order variables, and $\{x_i\}_{i < \omega}$ the list of first-order variables. Also let $\varphi(X_0)$ be a Π_k^1 -formula possibly with a unary predicate X_0 such that for any second-order X_i occurring in $\varphi(X_0)$, the first-order x_i does not occur in it.

For ordinals α and natural numbers $k \geq 0$, a $\Pi_k^1(\alpha)$ -formula (in the language $\{\in\}$, cf. Definition 4.4 for the class $\Pi_k^1(\alpha)$ in an expanded language) is a formula obtained from such a formula $\varphi(X_0)$ by replacing $X_0(t)$ by $t \in x_0 \wedge t \in \alpha$, replacing each second-order variable $X_i(t)$ occurring in the matrix of $\varphi(X_0)$ by $t \in x_i \wedge t \in \alpha$, restricting every second-order quantifier $\exists X_i, \forall X_i$ in φ to $\exists x_i \subset \alpha, \forall x_i \subset \alpha$, and restricting every unbounded quantifier $\exists z, \forall z$ in φ to $\exists z \in \alpha, \forall z \in \alpha$.

4. For ordinals α, β and a $\Pi_k^1(\alpha)$ -formula $\varphi \equiv \varphi(y \cap \alpha)$, $\varphi^{(\beta, \alpha)}$ denotes a $\Pi_k^1(\beta)$ -formula obtained from φ by replacing $z \in y \wedge z \in \alpha$ by $z \in y \wedge z \in \beta$, restricting every second-order quantifier $\exists x \subset \alpha, \forall x \subset \alpha$ in φ to $\exists x \subset \beta, \forall x \subset \beta$, and restricting every bounded quantifier $\exists z \in \alpha, \forall z \in \alpha$ in φ to $\exists z \in \beta, \forall z \in \beta$.

Definition 3.2 $T_N(\mathcal{K}, I, n)$ denotes the set theory defined as follows.

1. Its language is $\{\in, P, P_{I,n}, \text{Reg}, \mathcal{K}, \omega_1\}$ for a ternary predicate P , unary predicates $P_{I,n}$ and Reg , and individual constants \mathcal{K} and ω_1 .
2. Its axioms are obtained from those of Kripke-Platek set theory with the axiom of infinity $\text{KP}\omega$ in the expanded language, the axiom of constructibility, $V = L$ together with the following axiom schemata:

$$\begin{aligned}
(a) \quad & \text{Reg}(\omega_1), (\text{Reg}(\kappa) \rightarrow \kappa \in \text{ORD} \wedge (\kappa = \omega_1 \vee \kappa > \mathcal{K})), (\text{Reg}(\kappa) \rightarrow a \in \text{ORD} \cap \kappa \rightarrow \exists x, y \in \text{ORD} \cap \kappa [a < x \wedge P(\kappa, x, y)]), \text{ and } (P(\kappa, x, y) \rightarrow \{x, y\} \subset \text{ORD} \wedge x < y < \kappa \wedge \text{Reg}(\kappa)) \text{ and } (P(\kappa, x, y) \rightarrow a \in \text{ORD} \cap x \rightarrow \varphi[\kappa, a] \rightarrow \varphi^y[x, a]) \text{ for any } \Sigma_1\text{-formula } \varphi \text{ in the language } \{\in\}. \\
& (\forall x \in \text{ORD} \exists y [x \geq \mathcal{K} \rightarrow y > x \wedge \text{Reg}(y)]). \\
& (P_{I,n}(x) \rightarrow x \in \text{ORD}) \text{ and } (P_{I,n}(x) \rightarrow a \in \text{ORD} \cap x \rightarrow \varphi[a] \rightarrow \varphi^x[a]) \text{ for any } \Sigma_n\text{-formula } \varphi \text{ in the language } \{\in\}, \text{ and } (\mathcal{K} < a \in \text{ORD} \rightarrow \exists x \in \text{ORD} [a < x \wedge P_{I,n}(x)]).
\end{aligned}$$

$$(b) \quad (\mathcal{K} > \omega_1) \text{ and}$$

$$\forall x \subset \mathcal{K} [\varphi(x) \rightarrow \exists \rho < \mathcal{K} (\varphi^{(\rho, \mathcal{K})}(x \cap \rho))] \quad (5)$$

where $\varphi(x)$ is a $\Pi_N^1(\mathcal{K})$ -formula in the language $\{\in\}$.

Note that ‘ \mathcal{K} is regular’ , i.e., $\forall \alpha < \mathcal{K} \forall f \in {}^\alpha \mathcal{K} \exists \beta < \mathcal{K} (f''\alpha \subset \beta)$ follows from (5).

The following Lemma 3.3 is seen as in [3, 4].

Lemma 3.3 $T_N(\mathcal{K}, I) := \bigcup_{n \in \omega} T_N(\mathcal{K}, I, n)$ is equivalent to the set theory $\text{ZF} + (V = L) + (\mathcal{K} \text{ is } \Pi_N^1\text{-indescribable})$.

4 Operator controlled derivations for Π_N^1 -indescribable cardinals

In this section, operator controlled derivations are first introduced, and inferences $(\mathcal{K} \in M_N)$ for Π_N^1 -indescribability of \mathcal{K} are then eliminated from operator controlled derivations of Π_k^1 -sentences $\varphi^{V_{\mathcal{K}}}$ over \mathcal{K} .

In what follows n denotes a fixed positive integer. Let us write $\text{ZFL} := \text{ZF} + (V = L)$.

4.1 Intuitionistic fixed point theories $\text{FiX}^i(\text{ZFLK}_{k,n})$

For the fixed positive integer n and $0 \leq k < N$, let

$$\begin{aligned} b_n &:= \Psi_{\mathcal{K}^+, n}(\omega_{n-1}(I+1)), \quad a_n := \varphi(b_n)(b_n), \\ \gamma_{k,n} &:= \omega_{n-1}(I+1) + 1 + a_n(N-k), \quad \text{and} \\ \forall i < N-k &= lh(\vec{\alpha}_{k,n})(\vec{\alpha}_{k,n}(i) = \gamma_{k+i,n}) \end{aligned} \quad (6)$$

Then $\text{ZFLK}_{-1,n} := \text{ZFL} = \text{ZF} + (V = L)$, and for $N > k \geq 0$, $\text{ZFLK}_{k,n}, \text{ZFLK}_k$ denotes the set theories

$$\text{ZFLK}_{k,n} := \text{ZFL} + (\mathcal{K} \in Mh_{k,n}(\vec{\alpha}_{k,n})), \quad \text{ZFLK}_k := \bigcup \{ \text{ZFLK}_{k,n} : 0 < n < \omega \} \quad (7)$$

in the language $\{\in, \mathcal{K}, \omega_1\}$ with individual constants \mathcal{K}, ω_1 .

Let us also denote the set theory $\text{ZFL} + (\mathcal{K} \text{ is } \Pi_N^1\text{-indescribable})$ in the language $\{\in, \mathcal{K}, \omega_1\}$ by $\text{ZFLK}_N = \text{ZFLK}_{N,n}$ for any n .

To analyze the theory ZFLK_N , we need to handle a relation $(\mathcal{H}_{\gamma,n}, \Theta, \kappa, \text{ZFLK}_{k,n}) \vdash_b^a \Gamma$ defined in subsection 4.3, where n is the fixed integer, $k \leq N$, γ, a, b are codes of ordinals with $a < \omega_n(I+1)$, $b < I + \omega$, κ a regular cardinal and Θ are finite subsets of L and Γ a sequent, i.e., a finite set of sentences. As in [3, 4] the relation is defined for each $n < \omega$, as a fixed point,

$$H_n(k, \gamma, \Theta, a, b, \kappa, \Gamma) \Leftrightarrow (\mathcal{H}_{\gamma,n}, \Theta, \kappa, \text{ZFLK}_{k,n}) \vdash_b^a \Gamma \quad (8)$$

An intuitionistic fixed point theory $\text{FiX}^i(\text{ZFLK}_{k,n})$ over $\text{ZFLK}_{k,n}$ is introduced in [6], and shown to be a conservative extension of $\text{ZFLK}_{k,n}$.

Fix an X -strictly positive formula $\mathcal{Q}(X, x)$ in the language $\{\in, \mathcal{K}, \omega_1, =, X\}$ with an extra unary predicate symbol X . In $\mathcal{Q}(X, x)$ the predicate symbol X occurs only strictly positive. The language of $\text{FiX}^i(\text{ZFLK}_{k,n})$ is $\{\in, \mathcal{K}, =, \mathcal{Q}\}$ with a fresh unary predicate symbol \mathcal{Q} . The axioms in $\text{FiX}^i(\text{ZFLK}_{k,n})$ consist of the following:

1. All provable sentences in $\text{ZFLK}_{k,n}$ (in the language $\{\in, \mathcal{K}, \omega_1, =\}$).
2. Induction schema for any formula φ in $\{\in, \mathcal{K}, =, \mathcal{Q}\}$:

$$\forall x (\forall y \in x \varphi(y) \rightarrow \varphi(x)) \rightarrow \forall x \varphi(x) \quad (9)$$

3. Fixed point axiom:

$$\forall x[Q(x) \leftrightarrow \mathcal{Q}(Q, x)].$$

The underlying logic in $\text{FiX}^i(\text{ZFLK}_{k,n})$ is defined to be the intuitionistic logic.

(9) yields the following Lemma 4.1.

Lemma 4.1 *Let $<^\varepsilon$ denote a Δ_1 -predicate, which defines a well ordering of type ε_{I+1} . For each $m < \omega$ and each formula φ in $\{\in, \mathcal{K}, \omega_1, =, Q\}$,*

$$\text{FiX}^i(\text{ZFLK}_{k,n}) \vdash \forall x(\forall y <^\varepsilon x \varphi(y) \rightarrow \varphi(x)) \rightarrow \forall x <^\varepsilon \omega_m(I+1) \varphi(x).$$

The following Theorem 4.2 is shown in [6].

Theorem 4.2 *$\text{FiX}^i(\text{ZFLK}_{k,n})$ is a conservative extension of $\text{ZFLK}_{k,n}$ for each $k \geq -1$ and each $n \geq 0$.*

We will work in $\text{FiX}^i(\text{ZFL})$ throughout this section with a fixed integer n .

4.2 Classes of sentences

In this section we consider only the codes of ordinals less than $\omega_n(I+1)$ for a fixed positive integer n . $L_I := L = \bigcup_{\alpha \in ORD} L_\alpha$ denotes the universe with $\mathcal{K} \in L$. $\text{rk}_L(a) = \min\{\alpha \in ORD : a \in L_{\alpha+1}\}$ denotes the L -rank of a . Also set $\text{rk}_L(L_I) = I$.

The language \mathcal{L}_{cR} is obtained from the language $\{\in, P, P_{I,n}, \text{Reg}, \mathcal{K}, \omega_1\}$ by adding names (individual constants) c_a of each set $a \in L$, and for each regular cardinal $\kappa \leq \mathcal{K}$ and each subset $B \in \mathcal{P}(\mathcal{K}) \cap L_{\mathcal{K}+}$ adding *unary predicates* $R_{B,\kappa}$, their *complements* $\neg R_{B,\kappa}$ and unary predicate (second-order) variables $X_i, \neg X_i$ ($i \in \omega$). $R_{B,\kappa} [\neg R_{B,\kappa}]$ denotes the set $B \cap \kappa$ [the complement $\kappa \setminus B$], resp. c_a is identified with a , and predicate variables are denoted X, Y, \dots . The (individual) variables x, y, \dots and constants c_a are terms. Terms are denoted t, s, \dots .

Then *formulae* in \mathcal{L}_{cR} are constructed from *literals* $t \in s, t \notin s, P(t_1, t_2, t_3), \neg P(t_1, t_2, t_3), P_{I,n}(t), \neg P_{I,n}(t), \text{Reg}(t), \neg \text{Reg}(t), R_{B,\kappa}(t), \neg R_{B,\kappa}(t), X(t), \neg X(t)$ by propositional connectives \vee, \wedge , individual quantifiers $\exists x, \forall x$ and predicate (second-order) quantifiers $\exists X \subset \kappa, \forall X \subset \kappa$ for regular cardinals κ such that $\omega_1 < \kappa \leq \mathcal{K}$. Unbounded quantifiers $\exists x, \forall x$ are denoted by $\exists x \in L_I, \forall x \in L_I$, resp.

For formulae A in \mathcal{L}_{cR} , $\mathbf{k}(A)$ denotes the set of sets $a \equiv c_a$ occurring in A , but excluding subsets B in the predicates $R_{B,\kappa}$. $\mathbf{k}(A) \subset L_I$ is defined to include bounds of ‘bounded’ quantifiers and of ‘predicates’. Let us split $\mathbf{k}(A)$ in two sets $\mathbf{k}^E(A)$ and $\mathbf{k}^R(A)$. $\mathbf{k}^R(A)$ is the set of κ for which $R_{B,\kappa}$ occurs in A for some B . By definition we set $0 \in \mathbf{k}^R(A) \cap \mathbf{k}^E(A)$. In the following definition, Var denotes the set of variables.

Definition 4.3 1. $\mathbf{k}^E(\neg A) = \mathbf{k}^E(A)$ and similarly for \mathbf{k}^R .

2. $\mathbf{k}^R(M) = \{0\}$ for any literal M other than $R_{B,\kappa}(t), \neg R_{B,\kappa}(t)$.

3. $k^R(R_{B,\kappa}(t)) = \{\kappa, 0\}$.
4. $k^E(Q(t_1, \dots, t_m)) = \{t_1, \dots, t_m, 0\} \cap L_I$ for literals $Q(t_1, \dots, t_m)$ with predicates Q in the set $\{P, P_{I,n}, Reg, \in\} \cup \{X_i\}_{i \in \omega} \cup \{R_{B,\kappa} : B \in \mathcal{P}(\mathcal{K}) \cap L_{\mathcal{K}^+}, \kappa \leq \mathcal{K}\}$.
5. $k^E(A_0 \vee A_1) = k^E(A_0) \cup k^E(A_1)$ and similarly for k^R .
6. For $t \in L_I \cup \{L_I\} \cup Var$, $k^E(\exists x \in t A(x)) = (\{t\} \cap L_I) \cup k^E(A(x))$, while $k^R(\exists x \in t A(x)) = k^R(A(x))$.
7. $k^E(\exists X \subset \kappa A(X)) = \{\kappa\} \cup k^E(A(X))$, while $k^R(\exists X \subset \kappa A(X)) = k^R(A(X))$.
8. $k(A) = k^E(A) \cup k^R(A)$.

Definition 4.4 1. Δ_0 -formulae are constructed from literals $t \in s, t \notin s$, by propositional connectives \vee, \wedge , and *bounded* individual quantifiers $\exists x \in a, \forall x \in a$ ($a \in L_I$). Note that the predicates $P, P_{I,n}, Reg, R_{B,\kappa}, X$ do not occur in Δ_0 -formulae.

2. Putting $\Sigma_0 := \Pi_0 := \Delta_0$, the classes Σ_m and Π_m of formulae in the language $\{\in\}$ with terms are defined as usual using quantifiers $\exists x \in L_I, \forall x \in L_I$, where by definition $\Sigma_m \cup \Pi_m \subset \Sigma_{m+1} \cap \Pi_{m+1}$.

Each formula in $\Sigma_m \cup \Pi_m$ is in prenex normal form with alternating unbounded quantifiers and Δ_0 -matrix.

Note that the predicates $P, P_{I,n}, Reg, R_{B,\kappa}, X$ do not occur in Σ_m -formulae.

3. Let λ be a regular cardinal such that $\omega_1 < \lambda \leq \mathcal{K}$.
 $A \in \Delta_0(\lambda) = \Sigma_0(\lambda) = \Pi_0(\lambda)$ iff the sentence A contains no unbounded quantifiers $\exists x, \forall x, \max\{\text{rk}_L(t) : t \in k^R(A)\} \leq \lambda$ and $\max\{\text{rk}_L(t) : t \in k^E(A)\} < \lambda$, where $\text{rk}_L(x) := 0$ for any variable x .
 Note that the predicates $P, P_I, Reg, R_{B,\kappa}(\kappa \leq \lambda), X$ and predicate quantifiers $\exists X \subset \kappa, \forall X \subset \kappa$ ($\kappa < \lambda$) may occur in $\Delta_0(\lambda)$ -formulae.

4. $A \in \Sigma_1(\lambda)$ iff either $A \in \Delta_0(\lambda)$ or $A \equiv \exists x \in \lambda A_0$ with $A_0 \in \Delta_0(\lambda)$.
5. The class of sentences $\Sigma_m(\lambda), \Pi_m(\lambda)$ ($m < \omega$) are defined from $\Delta_0(\lambda)$ by applying bounded quantifiers $\exists x \in \lambda, \forall x \in \lambda$ as usual.
6. $\Pi_0^1(\lambda) := \bigcup_{m < \omega} \Sigma_m(\lambda) = \bigcup_{m < \omega} \Pi_m(\lambda)$.
7. $A \in \Sigma_1^1(\lambda)$ iff either $A \in \Pi_0^1(\lambda)$ or $A \equiv \exists X \subset \lambda B$ with $B \in \Pi_0^1(\lambda)$.
8. The class of sentences $\Sigma_m^1(\lambda), \Pi_m^1(\lambda)$ ($m < \omega$) are defined from $\Pi_0^1(\lambda)$ by applying predicate quantifiers $\exists X \subset \lambda, \forall X \subset \lambda$ as usual so that $\Sigma_m^1(\lambda) \cup \Pi_m^1(\lambda) \subset \Sigma_{m+1}^1(\lambda) \cap \Pi_{m+1}^1(\lambda)$.
 Set $\Pi_0^2(\lambda) := \bigcup_{m < \omega} \Pi_m^1(\lambda) = \bigcup_{m < \omega} \Sigma_m^1(\lambda)$.

9. (Cf. Definition 3.1.4.) For a $\Pi_k^1(\lambda)$ -sentence A and a regular cardinal κ with $\omega_1 < \kappa < \lambda$, $A^{(\kappa, \lambda)}$ denotes the result of replacing $R_{B, \lambda}(t)$ by $R_{B, \kappa}(t)$, restricting every second-order quantifier $\exists X \subset \lambda, \forall X \subset \lambda$ in A to $\exists X \subset \kappa, \forall X \subset \kappa$, and restricting every bounded quantifier $\exists z \in \lambda, \forall z \in \lambda$ in A to $\exists z \in \kappa, \forall z \in \kappa$.

If $\max\{\text{rk}_L(t) : t \in \mathbf{k}^R(A)\} \cap \lambda \leq \kappa$ and $\max\{\text{rk}_L(t) : t \in \mathbf{k}^E(A)\} < \kappa$, then $A^{(\kappa, \lambda)}$ is a $\Pi_k^1(\kappa)$ -sentence.

Also for a set Δ of $\Pi_k^1(\lambda)$ -sentences, $\Delta^{(\kappa, \lambda)} := \{A^{(\kappa, \lambda)} : A \in \Delta\}$.

In what follows we need to consider *sentences* A in the language \mathcal{L}_{cR} . Sentences are denoted A, C possibly with indices, while B, D denote sets in $\mathcal{P}(\mathcal{K}) \cap L_{\mathcal{K}^+}$.

Definition 4.5 A set $\Sigma^{\Sigma_{n+1}}(\lambda)$ of sentences is defined recursively as follows.

1. $\Sigma_{n+1} \subset \Sigma^{\Sigma_{n+1}}(\lambda)$.
2. Each literal including $\text{Reg}(a), P(a, b, c), P_{I, n}(a), R_{B, \kappa}(a)$ for $\kappa < \lambda$ and their negations is in $\Sigma^{\Sigma_{n+1}}(\lambda)$.
3. $\Sigma^{\Sigma_{n+1}}(\lambda)$ is closed under propositional connectives \vee, \wedge .
4. Suppose $\forall x \in b A(x) \notin \Delta_0$. Then $\forall x \in b A(x) \in \Sigma^{\Sigma_{n+1}}(\lambda)$ iff $A(\emptyset) \in \Sigma^{\Sigma_{n+1}}(\lambda)$ and $\text{rk}_L(b) < \lambda$.
5. Suppose $\exists x \in b A(x) \notin \Delta_0$. Then $\exists x \in b A(x) \in \Sigma^{\Sigma_{n+1}}(\lambda)$ iff $A(\emptyset) \in \Sigma^{\Sigma_{n+1}}(\lambda)$ and $\text{rk}_L(b) \leq \lambda$.
6. $\exists X \subset \kappa A(X) \in \Sigma^{\Sigma_{n+1}}(\lambda)$ iff $\forall X \subset \kappa A(X) \in \Sigma^{\Sigma_{n+1}}(\lambda)$ iff $A(\emptyset) \in \Sigma^{\Sigma_{n+1}}(\lambda)$ and $\kappa < \lambda$.

Note that if $\kappa < \lambda$, then any $\Pi_0^2(\kappa)$ -sentence is a $\Sigma^{\Sigma_{n+1}}(\lambda)$ -formula.

Definition 4.6 Let either $\lambda \in \text{Reg}$ and $x = \Psi_{\lambda, n}\beta$, or $x = \Psi_{I, n}\beta$ for some β . The domain $\text{dom}(F_{x \cup \{\lambda\}}^{\Sigma_1}) = \text{Hull}_{\Sigma_1}^I(x \cup \{\lambda\})$ of Mostowski collapse is extended to formulae.

$$\text{dom}(F_{x \cup \{\lambda\}}^{\Sigma_1}) = \{A \in \Sigma_1 \cup \Pi_1 : \mathbf{k}(A) \subset \text{Hull}_{\Sigma_1}^I(x \cup \{\lambda\})\}.$$

For $A \in \text{dom}(F_{x \cup \{\lambda\}}^{\Sigma_1})$, $F_{x \cup \{\lambda\}}^{\Sigma_1} A$ denotes the result of replacing each constant γ by $F_{x \cup \{\lambda\}}^{\Sigma_1}(\gamma)$, each unbounded existential quantifier $\exists z \in L_I$ by $\exists z \in L_{F_{x \cup \{\lambda\}}^{\Sigma_1}(I)}$, and each unbounded universal quantifier $\forall z \in L_I$ by $\forall z \in L_{F_{x \cup \{\lambda\}}^{\Sigma_1}(I)}$.

For sequent, i.e., finite set of sentences $\Gamma \subset \text{dom}(F_{x \cup \{\lambda\}}^{\Sigma_1})$, put $F_{x \cup \{\lambda\}}^{\Sigma_1} \Gamma = \{F_{x \cup \{\lambda\}}^{\Sigma_1} A : A \in \Gamma\}$.

Likewise the domain $\text{dom}(F_x^{\Sigma_n}) = \text{Hull}_{\Sigma_n}^I(x)$ is extended to

$$\text{dom}(F_x^{\Sigma_n}) = \{A \in \Sigma_n \cup \Pi_n : \mathbf{k}(A) \subset \text{Hull}_{\Sigma_n}^I(x)\}$$

and for formula $A \in \text{dom}(F_x^{\Sigma_n})$, $F_x^{\Sigma_n} A$, and sequent $\Gamma \subset \text{dom}(F_x^{\Sigma_n})$, $F_x^{\Sigma_n} \Gamma$ are defined similarly.

Proposition 4.7 Suppose that either $\lambda \in \text{Reg}$ and $x = \Psi_{\lambda,n}\beta$, or $x = \Psi_{I,n}\beta$ for some β . For $F = F_{x \cup \{\lambda\}}^{\Sigma_1}, F_y^{\Sigma_n}$ and $A \in \text{dom}(F)$, $L_I \models A \leftrightarrow F''A$.

Proof. Note that the predicates $P, P_{I,n}, \text{Reg}, R_{B,\kappa}, X$ do not occur in Σ_m -formulae. \square

The assignment of disjunctions and conjunctions to sentences is defined as in [4] slightly modified, and by adding the clauses for second-order formulae.

Let us define truth values of literals.

1. For any literal M , $\neg M$ is *true* iff M is not true.
2. $a \in b$ is *true* iff $L \models a \in b$.
3. $R_{B,\kappa}(a)$ is *true* iff $a \in B \cap \kappa$.
4. $\text{Reg}(a)$ is *true* iff $a \in \text{Reg}$, i.e., either $a = \omega_1$ or $\mathcal{K} < a < I$ is regular.
5. $P(a, b, c)$ is *true* iff $a \in \text{Reg}$ and $\exists \alpha < \omega_n(I+1)[b = \Psi_{a,n}\alpha \ \& \ c = F_{b \cup \{a\}}^{\Sigma_1}(I)]$.
6. $P_{I,n}(a)$ is *true* iff $\exists \alpha < \omega_n(I+1)[a = \Psi_{I,n}\alpha]$.

Definition 4.8 1. If M is a literal, then for $J := 0$

$$M := \begin{cases} \bigvee (A_\iota)_{\iota \in J} & \text{if } M \text{ is false (in } L_I) \\ \bigwedge (A_\iota)_{\iota \in J} & \text{if } M \text{ is true} \end{cases}$$

2. $(A_0 \vee A_1) := \bigvee (A_\iota)_{\iota \in J}$ and $(A_0 \wedge A_1) := \bigwedge (A_\iota)_{\iota \in J}$ for $J := 2$.

3. Let $(\exists z \in b \theta[z]) \in \Sigma_n$ for $b \in L_I \cup \{L_I\}$. Then for the set

$$d := \mu z \in b \theta[z] := \min_{<_L} \{d : (d \in b \wedge \theta[d]) \vee (\neg \exists z \in b \theta[z] \wedge d = 0)\} \quad (10)$$

with a canonical well ordering $<_L$ on L , and $J = \{d\}$

$$\begin{aligned} \exists z \in b \theta[z] &:= \bigvee (d \in b \wedge \theta[d])_{d \in J} \\ \forall z \in b \neg \theta[z] &:= \bigwedge (d \in b \rightarrow \neg \theta[d])_{d \in J} \end{aligned} \quad (11)$$

where $d \in b$ denotes a true literal, e.g., $d \notin d$ when $b = L_I$.

This case is applied only when $\exists z \in b \theta[z]$ is a formula in $\{\in\} \cup L_I$, and $(\exists z \in b \theta[z]) \in \Sigma_n$.

4. For a sentence $\exists X \subset \kappa \theta[X]$ with a regular $\omega_1 < \kappa \leq \mathcal{K}$ set $J := \mathcal{P}(\mathcal{K}) \cap L_{\mathcal{K}+}$ and let

$$\exists X \subset \kappa \theta[X] := \bigvee (\theta[R_{B,\kappa}])_{B \in J} \text{ and } \forall X \subset \kappa \theta[X] := \bigwedge (\theta[R_{B,\kappa}])_{B \in J}$$

where $\theta[R_{B,\kappa}]$ is obtained from $\theta[X]$ by replacing $X(t)$ [$\neg X(t)$] by $R_{B,\kappa}(t)$ [$\neg R_{B,\kappa}(t)$], resp.

5. Otherwise set for $a \in L_I \cup \{L_I\}$ and $J := \{b : b \in a\}$

$$\exists x \in a A(x) := \bigvee (A(b))_{b \in J} \text{ and } \forall x \in a A(x) := \bigwedge (A(b))_{b \in J}.$$

This case is applied if one of the predicates $P, P_{I,n}, Reg, R_{B,\kappa}$ occurs in $\exists x \in a A(x)$, or $(\exists x \in a A(x)) \notin \Sigma_n$.

The definition of the rank $\text{rk}(A)$ of sentences A is modified from [3,4] so as to the following Propositions 4.10.4 and 4.10.6 holds. The rank $\text{rk}(A)$ of sentences A is defined by recursion on the number of symbols occurring in A .

Definition 4.9 1. $\text{rk}(\neg A) := \text{rk}(A)$.

2. $\text{rk}(M) := \max\{\text{rk}_L(t) : t \in \mathbf{k}(M)\}$ for any (closed) literal M .

3. $\text{rk}(A_0 \vee A_1) := \max\{\text{rk}(A_0), \text{rk}(A_1)\} + 1$.

4. (a) $\text{rk}(\exists x < I[b < x \wedge P_{I,n}(x)]) := I$.

(b) $\text{rk}(\exists x < \lambda \exists y < \lambda[b < x \wedge P(\lambda, x, y)]) := \max\{\lambda + 1, \text{rk}_L(b)\}$ for $\lambda \in Reg$.

(c) $\text{rk}(\exists x \in a A(x)) := \max\{\omega \text{rk}_L(a), \text{rk}(A(\emptyset)) + 1\}$ in other cases.

5. $\text{rk}(\exists X \subset \kappa A(X)) := \max\{\kappa, \text{rk}(A(R_{\emptyset, \kappa})) + 1\}$.

Proposition 4.10 Let A be a sentence with $A \simeq \bigvee (A_\iota)_{\iota \in J}$ or $A \simeq \bigwedge (A_\iota)_{\iota \in J}$.

1. $A \in \Sigma^{\Sigma_{n+1}}(\lambda) \Rightarrow \forall \iota \in J (A_\iota \in \Sigma^{\Sigma_{n+1}}(\lambda))$.

2. For an ordinal $\lambda \leq I$ with $\omega\lambda = \lambda$, $\text{rk}(A) < \lambda \Rightarrow A \in \Sigma^{\Sigma_{n+1}}(\lambda)$.

3. $\text{rk}(A) < I + \omega$.

4. $\max\{\text{rk}_L(t) : t \in \mathbf{k}(A)\} \leq \text{rk}(A) \in \{\omega \text{rk}_L(a) + i : a \in \mathbf{k}(A) \cup \{I\}, i \in \omega\} \subset \text{Hull}_{\Sigma_1}(\mathbf{k}(A))$.

5. $\forall \iota \in J (\text{rk}(A_\iota) < \text{rk}(A))$ if A is not a formula of the form $\exists x < \lambda \exists y < \lambda[b < x \wedge P(\lambda, x, y)]$ for some $\lambda \in Reg$.

6. For an ordinal $\lambda \leq I$ with $\omega\lambda = \lambda$, $\text{rk}(A) < \lambda \Rightarrow A \in \Delta_0(\lambda)$

4.3 Operator controlled derivations

Definition 4.11 By an *operator* we mean a map $\mathcal{H}, \mathcal{H} : \mathcal{P}(L_I) \rightarrow \mathcal{P}(L_I \cup \omega_{n+1}(I + 1))$, such that

1. $\forall X \subset L_I [X \subset \mathcal{H}(X)]$.

2. $\forall X, Y \subset L_I [Y \subset \mathcal{H}(X) \Rightarrow \mathcal{H}(Y) \subset \mathcal{H}(X)]$.

For an operator \mathcal{H} and $\Theta, \Lambda \subset L_I$, $\mathcal{H}[\Theta](X) := \mathcal{H}(X \cup \Theta)$, and $\mathcal{H}[\Theta][\Lambda] := (\mathcal{H}[\Theta])[\Lambda]$, i.e., $\mathcal{H}[\Theta][\Lambda](X) = \mathcal{H}(X \cup \Theta \cup \Lambda)$.

Obviously $\mathcal{H}_{\alpha,n}$ is an operator for any α, n , and if \mathcal{H} is an operator, then so is $\mathcal{H}[\Theta]$.

Sequents are finite sets of sentences, and inference rules are formulated in one-sided sequent calculus. Let $\mathcal{H} = \mathcal{H}_{\gamma,n}$ be an operator, Θ a finite set of subsets of \mathcal{K} , $\sigma \leq I$ a cardinal, Γ a sequent, $-2 \leq k \leq N$, $a < \omega_n(I+1)$ and $b < I + \omega$.

We define a relation $(\mathcal{H}, \Theta, \sigma, \text{ZFLK}_{k,n}) \vdash_b^a \Gamma$, which is read ‘there exists an infinitary derivation of Γ which is $(\Theta, \sigma, \text{ZFLK}_{k,n})$ -controlled by \mathcal{H} , and whose height is at most a and its cut rank is less than b ’.

Recall that $\kappa \in \text{Reg}$ iff either $\kappa > \mathcal{K}$ is regular or $\kappa = \omega_1$. $\kappa, \lambda, \sigma, \pi$ ranges over $\text{Reg}^+ := \text{Reg} \cup \{I\}$.

Definition 4.12 $(\mathcal{H}, \Theta, \sigma, \text{ZFLK}_{k,n}) \vdash_b^a \Gamma$ holds if

$$\text{k}(\Gamma) \cup \{a\} \subset \mathcal{H}[\Theta] = \mathcal{H}(\Theta) \quad (12)$$

and one of the following cases holds:

(\bigvee) $A \simeq \bigvee \{A_\iota : \iota \in J\}$, $A \in \Gamma$ and there exist $\iota \in J$ and $a(\iota) < a$ such that

$$\text{rk}_L(\iota) < \sigma \Rightarrow \text{rk}_L(\iota) < a \quad (13)$$

and $(\mathcal{H}, \Theta, \sigma, \text{ZFLK}_{k,n}) \vdash_b^{a(\iota)} \Gamma, A_\iota$.

(\bigwedge) $A \simeq \bigwedge \{A_\iota : \iota \in J\}$, $A \in \Gamma$ and for every $\iota \in J$ there exists an $a(\iota) < a$ such that $(\mathcal{H}, \Theta \cup \{\iota\}, \sigma, \text{ZFLK}_{k,n}) \vdash_b^{a(\iota)} \Gamma, A_\iota$.

(*cut*) There exist $a_0 < a$ and C such that $\text{rk}(C) < b$ and $(\mathcal{H}, \Theta, \sigma, \text{ZFLK}_{k,n}) \vdash_b^{a_0} \Gamma, \neg C$ and $(\mathcal{H}, \Theta, \sigma, \text{ZFLK}_{k,n}) \vdash_b^{a_0} C, \Gamma$.

(P_λ) $\lambda \in \text{Reg}$ and there exists $\alpha < \lambda$ such that $(\exists x < \lambda \exists y < \lambda [\alpha < x \wedge P(\lambda, x, y)]) \in \Gamma$.

($\text{F}_{x \cup \{\lambda\}}^{\Sigma_1}$) $\lambda \in \text{Reg}$, $x = \Psi_{\lambda,n} \beta \in \mathcal{H}$ for a β and there exist $a_0 < a$, $\Gamma_0 \subset \Sigma_1$ and Λ such that $\text{k}(\Gamma_0) \subset \text{Hull}_{\Sigma_1}^I((\mathcal{H} \cap x) \cup \{\lambda\})$, $\Gamma = \Lambda \cup (F_{x \cup \{\lambda\}}^{\Sigma_1} \text{''} \Gamma_0)$ and $(\mathcal{H}, \Theta, \sigma, \text{ZFLK}_{k,n}) \vdash_b^{a_0} \Lambda, \Gamma_0$, where $F_{x \cup \{\lambda\}}^{\Sigma_1}$ denotes the Mostowski collapse $F_{x \cup \{\lambda\}}^{\Sigma_1} : \text{Hull}_{\Sigma_1}^I(x \cup \{\lambda\}) \leftrightarrow L_{F_{x \cup \{\lambda\}}^{\Sigma_1}(I)}$.

($\text{P}_{I,n}$) There exists $\alpha < I$ such that $(\exists x < I [\alpha < x \wedge P_{I,n}(x)]) \in \Gamma$.

($\text{F}_x^{\Sigma_n}$) $x = \Psi_{I,n} \beta \in \mathcal{H}$ for a β and there exist $a_0 < a$, $\Gamma_0 \subset \Sigma_n$ and Λ such that $\text{k}(\Gamma_0) \subset \text{Hull}_{\Sigma_n}^I(\mathcal{H} \cap x)$, $\Gamma = \Lambda \cup (F_x^{\Sigma_n} \text{''} \Gamma_0)$ and $(\mathcal{H}, \Theta, \sigma, \text{ZFLK}_{k,n}) \vdash_b^{a_0} \Lambda, \Gamma_0$, where $F_x^{\Sigma_n}$ denotes the Mostowski collapse $F_x^{\Sigma_n} : \text{Hull}_{\Sigma_n}^I(x) \leftrightarrow L_{F_x^{\Sigma_n}(I)}$.

$(\exists^2(\kappa))$ ($\omega_1 < \kappa \leq \mathcal{K}$)

$\exists X \subset \kappa \theta[X] \simeq \bigvee (\theta[R_{B,\kappa}])_{B \in J}$, $(\exists X \subset \kappa \theta[X]) \in \Gamma \cap \Pi_0^2(\kappa)$ and there exist $B \in J = \mathcal{P}(\mathcal{K}) \cap L_{\mathcal{K}^+}$ and $a_0 < a$ such that $(\mathcal{H}, \Theta, \sigma, \text{ZFLK}_{k,n}) \vdash_b^{a_0} \Gamma, \theta[R_{B,\kappa}]$.

$(\forall^2(\kappa))$ ($\omega_1 < \kappa \leq \mathcal{K}$)

$\forall X \subset \kappa \theta[X] \simeq \bigwedge (\theta[R_{B,\kappa}])_{B \in J}$, $(\forall X \subset \kappa \theta[X]) \in \Gamma \cap \Pi_0^2(\kappa)$ and there exists an $a_0 < a$ such that for any $B \in J = \mathcal{P}(\mathcal{K}) \cap L_{\mathcal{K}^+}$ $(\mathcal{H}, \Theta, \sigma, \text{ZFLK}_{k,n}) \vdash_b^{a_0} \Gamma, \theta[R_{B,\kappa}]$.

Note that the ordinal a_0 is independent from $B \in J$.

$(\pi \in Mh_{i,n}(\vec{\alpha})[\Theta], \vec{\nu})$ *This inference rule is only for $k \geq -1$.*

There exist a regular cardinal $\pi < \mathcal{K}$, a number $i < N$, sequences of ordinals $\vec{\alpha} = (\alpha_0, \dots, \alpha_{N-i-1})$, $\vec{\nu} = (\nu_0, \dots, \nu_{N-i-1})$ such that $lh(\vec{\alpha}) = lh(\vec{\nu}) = N - i$. Moreover there are ordinals $a_\ell, a_r(\rho), a_0$, and a finite set Δ of $\Sigma_i^1(\pi)$ -sentences enjoying the following conditions:

1. $\vec{\nu} \in \mathcal{H}_{\vec{\nu},n}[\Theta] \cap \vec{\alpha}(\Leftrightarrow \forall j < lh(\vec{\nu})(\nu_j \in \mathcal{H}_{\nu_j,n}[\Theta] \cap \alpha_j))$, $\pi \in Mh_{i,n}(\vec{\alpha})[\Theta]$, and $b \geq \pi$.

2.

$$\pi \in \mathcal{H}[\Theta] \quad (14)$$

3.

$$\forall a \in K(\vec{\alpha})(a \leq \gamma) \quad (15)$$

where $\mathcal{H} = \mathcal{H}_{\gamma,n}$.

4. For each $\delta \in \Delta$

$$(\mathcal{H}, \Theta, \sigma, \text{ZFLK}_{k,n}) \vdash_b^{a_\ell} \Gamma, \neg \delta$$

5. Let $H_i(\vec{\nu}, \gamma)$ denote the *resolvent class* for $\pi \in Mh_{i,n}(\vec{\alpha})[\Theta]$ with respect to $\vec{\nu}$ and γ :

$$H_i(\vec{\nu}, \gamma) := \left\{ \rho \in \bigcap_{j < lh(\vec{\alpha})} Mh_{i+j,n}(((\vec{\nu} \bullet \vec{\alpha})[j])[\Theta] \cap \pi : \mathcal{H}_{\gamma,n}[\Theta](\rho) \cap \pi \subset \rho) \right\} \quad (16)$$

where $\mathcal{H} = \mathcal{H}_{\gamma,n}$. Then for any $\rho \in H_i(\vec{\nu}, \gamma)$

$$(\mathcal{H}, \Theta \cup \{\rho\}, \sigma, \text{ZFLK}_{k,n}) \vdash_b^{a_r(\rho)} \Gamma, \Delta^{(\rho, \pi)}$$

6.

$$\sup\{a_\ell, a_r(\rho) : \rho \in H_k(\vec{\nu}, \gamma)\} \leq a_0 \in \mathcal{H}[\Theta] \cap a \quad (17)$$

$(\mathcal{K} \in Mh_{k,n}(\vec{\alpha})[\Theta], \vec{\nu})$ *This inference rule is only for $k \geq 0$.*

There are sequences of ordinals $\vec{\alpha} = (\alpha_0, \dots, \alpha_{N-k-1})$, $\vec{\nu} = (\nu_0, \dots, \nu_{N-k-1})$ such that $lh(\vec{\alpha}) = lh(\vec{\nu}) = N - k$. Moreover there are ordinals $a_\ell, a_r(\rho), a_0$, and a finite set Δ of $\Sigma_k^1(\mathcal{K})$ -sentences enjoying the following conditions:

1. $\vec{\nu} \in \mathcal{H}_{\vec{\nu},n}[\Theta] \cap \vec{\alpha}$ and $b \geq \mathcal{K}$.

Note that $\mathcal{K} \in Mh_{k,n}(\vec{\alpha})[\Theta]$ is *not required* here.

2.

$$\forall i < lh(\vec{\alpha})(\vec{\alpha}(i) \leq \min\{\gamma_{k+i,n}, \gamma\}) \quad (15)$$

where $\mathcal{H} = \mathcal{H}_{\gamma,n}$ and $\vec{\alpha}_{k,n}(i) = \gamma_{k+i,n}$, cf. (6) for the ordinal $\gamma_{k,n}$.

3. For each $\delta \in \Delta$

$$(\mathcal{H}, \Theta, \sigma, \text{ZFLK}_{k,n}) \vdash_b^{a_\ell} \Gamma, \neg\delta$$

4. Let $H_k(\vec{\nu}, \gamma)$ denote the *resolvent class* for $\mathcal{K} \in Mh_{k,n}(\vec{\alpha})[\Theta]$ with respect to $\vec{\nu}$ and γ :

$$H_k(\vec{\nu}, \gamma) := \{\rho \in \bigcap_{i < lh(\vec{\alpha})} Mh_{k+i,n}((\vec{\nu} \bullet \vec{\alpha})[i])[\Theta] \cap \mathcal{K} : \mathcal{H}_{\gamma,n}[\Theta](\rho) \cap \mathcal{K} \subset \rho\} \quad (16)$$

where $\mathcal{H} = \mathcal{H}_{\gamma,n}$.

Then for any $\rho \in H_k(\vec{\nu}, \gamma)$

$$(\mathcal{H}, \Theta \cup \{\rho\}, \sigma, \text{ZFKL}_{k,n}) \vdash_b^{a_r(\rho)} \Gamma, \Delta^{(\rho, \mathcal{K})}$$

In particle when $k = N$,

$$H_N(\gamma) := H_N(\vec{\nu}, \gamma) = \{\rho < \mathcal{K} : \rho \text{ is regular and } \mathcal{H}_{\gamma,n}[\Theta](\rho) \cap \mathcal{K} \subset \rho\}$$

5.

$$\sup\{a_\ell, a_r(\rho) : \rho \in H_k(\vec{\nu}, \gamma)\} \leq a_0 \in \mathcal{H}[\Theta] \cap a \quad (17)$$

Some comments on the inference rule $(\pi \in Mh_{i,n}(\vec{\alpha})[\Theta], \vec{\nu})$ are helpful.

The inference rule $(\pi \in Mh_{i,n}(\vec{\alpha})[\Theta], \vec{\nu})$ says that $H_i(\vec{\nu}, \gamma)$ is Π_i^1 -indescribable in π , and is depicted as follows by suppressing $\sigma, \text{ZFLK}_{k,n}$:

$$\frac{\{(\mathcal{H}, \Theta) \vdash_b^{a_\ell} \Gamma, \neg\delta\}_{\delta \in \Delta} \quad \{(\mathcal{H}, \Theta \cup \{\rho\}) \vdash_b^{a_r(\rho)} \Gamma, \Delta^{(\rho, \pi)}\}_{\rho \in H_i(\vec{\nu}, \gamma) \cap \pi}}{(\mathcal{H}, \Theta) \vdash_b^a \Gamma} \quad (\pi \in Mh_{i,n}(\vec{\alpha})[\Theta], \vec{\nu})$$

The inference rule $(\mathcal{K} \in Mh_{N,n}(\vec{\alpha})[\Theta])$ is denoted by $(\mathcal{K} \in M_N)$, and depicted as follows:

$$\frac{\{(\mathcal{H}, \Theta) \vdash_b^{a_\ell} \Gamma, \neg\delta\}_{\delta \in \Delta} \quad \{(\mathcal{H}, \Theta \cup \{\rho\}) \vdash_b^{a_r(\rho)} \Gamma, \Delta^{(\rho, \mathcal{K})}\}_{\rho \in H_N(\gamma)}}{(\mathcal{H}, \Theta) \vdash_b^a \Gamma} \quad (\mathcal{K} \in M_N)$$

where Δ is a finite set of Σ_N^1 -sentences.

Note that there occurs only the inference rules $(\mathcal{K} \in Mh_{k,n}(\vec{\alpha})[\Theta], \vec{\nu})$ in the derivation establishing the fact $(\mathcal{H}, \Theta, \kappa, \text{ZFLK}_{k,n}) \vdash_b^a \Gamma$ for $k \geq 0$. In particular there occurs no such inference rules for any $k \geq 0$ in the derivation establishing the fact $(\mathcal{H}, \Theta, \kappa, \text{ZFLK}_{-1,n}) \vdash_b^a \Gamma$.

When an inference rule $(\pi \in Mh_{i,n}(\vec{\alpha})[\Theta], \vec{\nu})$ is applied for $\pi < \mathcal{K}$, $\pi \in Mh_{i,n}(\vec{\alpha})[\Theta]$ has to be enjoyed. As contrasted with this case $\pi < \mathcal{K}$, we don't assume that $\mathcal{K} \in Mh_{k,n}(\vec{\alpha})[\Theta]$ holds in applying inference rules $(\mathcal{K} \in Mh_{k,n}(\vec{\alpha})[\Theta], \vec{\nu})$. Therefore each inference rule in the derivation establishing the fact $(\mathcal{H}, \Theta, \kappa, \text{ZFLK}_{-1,n}) \vdash_b^a \Gamma$ is *correct* demonstrably in ZFL in the sense that if (the disjunction of) each upper sequent holds, then so is its lower sequent. Moreover for $k \geq 0$ and $\Theta \subset (\mathcal{K} + 1)$ each inference rule $(\mathcal{K} \in Mh_{k,n}(\vec{\alpha})[\Theta], \vec{\nu})$ is correct demonstrably in $\text{ZFLK}_{k,n}$, cf. Proposition 2.5.5 and (7).

An inspection to Definition 4.12 shows that there exists a strictly positive formula H_n such that the relation $(\mathcal{H}_{\gamma,n}, \Theta, \kappa, \text{ZFLK}_{k,n}) \vdash_b^a \Gamma$ is a fixed point of H_n as in (8).

In what follows the relation should be understood as a fixed point of H_n , and recall that we are working in the intuitionistic fixed point theory $\text{Fix}^i(\text{ZFL})$ over ZFL defined in subsection 4.1.

We will state some lemmata for the operator controlled derivations with sketches of their proofs since these can be shown as in [4, 9].

In what follows by an operator \mathcal{H} we mean an $\mathcal{H}_{\gamma,n}$ for an ordinal γ . Also except otherwise stated, $(\mathcal{H}, \Theta) \vdash_b^a \Gamma$ [$(\mathcal{H}, \Theta, \kappa) \vdash_b^a \Gamma$] denotes $(\mathcal{H}, \Theta, \kappa, \text{ZFLK}_{k,n}) \vdash_b^a \Gamma$ for some arbitrarily fixed κ and k, n [for some fixed arbitrarily k, n], resp.

Lemma 4.13 (Inversion lemma for predicate quantifiers)

Let $(\forall X \subset \pi \theta[X]) \in \Pi_0^2(\pi)$ and assume $(\mathcal{H}, \Theta) \vdash_b^a \Gamma, \forall X \subset \pi \theta[X]$. Then for any subset $B \in \mathcal{P}(\mathcal{K}) \cap L_{\mathcal{K}+}$, $(\mathcal{H}, \Theta) \vdash_b^a \Gamma, \theta[R_{B,\pi}]$.

Lemma 4.14 (Tautology) $(\mathcal{H}, k(\Gamma \cup \{A\}), I) \vdash_0^{2\text{rk}(A)} \Gamma, \neg A, A$.

To interpret the axiom of Π_N^1 -indescribability of \mathcal{K}

$$\forall x \subset \mathcal{K} [\varphi(x) \rightarrow \exists \rho < \mathcal{K} (\varphi^{(\rho, \mathcal{K})}(x \cap \rho))] \quad (5)$$

we need to show the equivalence of $\Pi_k^1(\kappa)$ -formula and its translation in the first-order language $\{\in\}$.

Let $\varphi(X)$ be a Π_N^1 -sentence possibly with a predicate constant X in the language $\{\in\} \cup \{X_i\}_{i < \omega}$, $\varphi_0(x)$ its $\Pi_N^1(\mathcal{K})$ -translation in $\{\in\}$ defined in Definition 3.1.3, and $\varphi_1(X)$ denote the $\Pi_N^1(\mathcal{K})$ -sentence obtained from $\varphi(X)$ by restricting second-order quantifiers $\exists Y, \forall Y$ to $\exists Y \subset \mathcal{K}, \forall Y \subset \mathcal{K}$, and restricting first-order (unbounded) quantifiers $\exists z, \forall z$ to $\exists z \in \mathcal{K}, \forall z \in \mathcal{K}$.

Let us temporarily introduce a complexity measure $d(\varphi) < \omega$ of second-order formulae φ in $\{\in\} \cup \{X_i\}_i$. $d(x \in y) = d(X(y)) = 0$, $d(\varphi_0 \vee \varphi_1) = \max\{d(\varphi_i) : i < 2\} + 1$, $d(\exists x \varphi) = d(\varphi) + 1$, $d(\exists X \varphi) = d(\varphi) + 2$ and $d(\neg \varphi) = d(\varphi)$.

Proposition 4.15 For any set $B \in \mathcal{P}(\mathcal{K}) \cap L_{\mathcal{K}+}$, any $\rho \leq \mathcal{K}$ and any Γ

$$(\mathcal{H}, k(\Gamma) \cup \{B, \rho\}, I) \vdash_0^{2d(\varphi)} \Gamma, \neg \varphi_0^{(\rho, \mathcal{K})}(B \cap \rho), \varphi_1^{(\rho, \mathcal{K})}(R_{B,\rho})$$

and

$$(\mathcal{H}, k(\Gamma) \cup \{B, \rho\}, I) \vdash_0^{2d(\varphi)} \Gamma, \varphi_0^{(\rho, \mathcal{K})}(B \cap \rho), \neg \varphi_1^{(\rho, \mathcal{K})}(R_{B,\rho})$$

Proof. This is seen by induction on $d(\varphi)$. Let us check one half of the case when $\varphi(X) \equiv (\exists Y \theta(Y, X))$. Let $\Theta = k(\Gamma) \cup \{B, \rho\}$ and $d = d(\theta)$. By IH we have for any $D \subset \mathcal{K}$

$$(\mathcal{H}, \Theta \cup \{D\}, I) \vdash_0^{2d} \Gamma, \neg \theta_0^{(\rho, \mathcal{K})}(D \cap \rho, B \cap \rho), \theta_1^{(\rho, \mathcal{K})}(R_{D, \rho}, R_{B, \rho})$$

Hence for any $D \subset \rho$

$$(\mathcal{H}, \Theta \cup \{D\}, I) \vdash_0^{2d+2} \Gamma, (D \not\subset \rho) \vee \neg \theta_0^{(\rho, \mathcal{K})}(D, B \cap \rho), \exists Y \subset \rho \theta_1^{(\rho, \mathcal{K})}(Y, R_{B, \rho})$$

On the other side, if $a \not\subset \rho$, then $(\mathcal{H}, \Theta \cup \{a\}, I) \vdash_0^1 a \not\subset \rho$. Hence

$$(\mathcal{H}, \Theta \cup \{a\}, I) \vdash_0^2 (a \not\subset \rho) \vee \neg \theta_0^{(\rho, \mathcal{K})}(a, B \cap \rho)$$

By a (\wedge) for $\neg \varphi_0^{(\rho, \mathcal{K})}(B \cap \rho) \equiv \forall y(y \subset \rho \rightarrow \neg \theta_0^{(\rho, \mathcal{K})}(y, B \cap \rho))$ we obtain

$$(\mathcal{H}, \Theta, I) \vdash_0^{2d+3} \Gamma, \forall y(y \subset \rho \rightarrow \neg \theta_0^{(\rho, \mathcal{K})}(y, B \cap \rho)), \exists Y \subset \rho \theta_1^{(\rho, \mathcal{K})}(Y, R_{B, \rho})$$

□

Definition 4.16 For a formula $\exists x \in d A$ and ordinals $\lambda = \text{rk}_L(d) \in \text{Reg} \cup \{I\}$, $\alpha, (\exists x \in d A)^{(\exists \lambda | \alpha)}$ denotes the result of restricting the *outermost existential quantifier* $\exists x \in d$ to $\exists x \in L_\alpha$, $(\exists x \in d A)^{(\exists \lambda | \alpha)} \equiv (\exists x \in L_\alpha A)$.

In what follows $F_{x, \lambda}$ denotes $F_{x, \lambda}^{\Sigma_1}$ when $\lambda \in R$, and $F_x^{\Sigma_n}$ when $\lambda = I$.

Lemma 4.17 (Boundedness)

Let $\lambda \in \text{Reg} \cup \{I\}$, $C \equiv (\exists x \in d A)$ and $C \notin \{\exists x < \lambda \exists y < \lambda [\alpha < x \wedge P(\lambda, x, y)] : \alpha < \lambda \in \text{Reg}\} \cup \{\exists x < I [\alpha < x \wedge P_{I, n}(x)] : \alpha < I\}$. Assume that $\text{rk}(C) = \lambda = \text{rk}_L(d)$ and C is not a second-order formula.

1.

$$(\mathcal{H}, \Theta, \lambda) \vdash_c^a \Lambda, C \ \& \ a \leq b \in \mathcal{H} \cap \lambda \Rightarrow (\mathcal{H}, \Theta, \lambda) \vdash_c^a \Lambda, C^{(\exists \lambda | b)}.$$

2.

$$(\mathcal{H}, \Theta, \lambda) \vdash_c^a \Lambda, \neg C \ \& \ b \in \mathcal{H} \cap \lambda \Rightarrow (\mathcal{H}, \Theta, \lambda) \vdash_c^a \Lambda, \neg(C^{(\exists \lambda | b)}).$$

In the following Lemma 4.18, note that $\text{rk}(\exists x < \lambda \exists y < \lambda [\alpha < x \wedge P(\lambda, x, y)]) = \lambda + 1$ for $\alpha < \lambda \in \text{Reg}$, and $\text{rk}(\exists x < I [\alpha < x \wedge P_{I, n}(x)]) = I$.

Lemma 4.18 (Predicative Cut-elimination)

1. Suppose $[c, c + \omega^a[\cap(\{\lambda + 1 : \lambda \in \text{Reg}\} \cup \{I\}) = \emptyset \text{ and } k \neq -2 \Rightarrow] c, c + \omega^a] \cap \{\kappa \leq \mathcal{K} : \omega_1 < \kappa \text{ is regular}\} = \emptyset$. Then $(\mathcal{H}, \Theta, \kappa, \text{ZFLK}_{k, n}) \vdash_{c+\omega^a}^b \Gamma \ \& \ a \in \mathcal{H}[\Theta] \Rightarrow (\mathcal{H}, \Theta, \kappa, \text{ZFLK}_{k, n}) \vdash_c^{\varphi^{ab}} \Gamma$.
2. For $\lambda \in \text{Reg}$, $(\mathcal{H}_\gamma, \Theta, \kappa, \text{ZFLK}_{k, n}) \vdash_{\lambda+2}^b \Gamma \ \& \ \gamma \in \mathcal{H}_\gamma \ \& \Rightarrow (\mathcal{H}_{\gamma+b}, \Theta, \kappa, \text{ZFLK}_{k, n}) \vdash_{\lambda+1}^{\omega^b} \Gamma$.

3. $(\mathcal{H}_\gamma, \Theta, \kappa, \text{ZFLK}_{k,n}) \vdash_{I+1}^b \Gamma \ \& \ \gamma \in \mathcal{H}_\gamma \ \& \Rightarrow (\mathcal{H}_{\gamma+b}, \Theta, \kappa, \text{ZFLK}_{k,n}) \vdash_I^{\omega^b} \Gamma.$

The following Lemma 4.19 is seen as in Lemma 4.10 of [3].

Lemma 4.19 (Collapsing)

Assume $\mathcal{K} < \lambda \leq \sigma \in \{\omega_\alpha : \alpha \leq I\} \ \& \ \lambda \in \text{Reg}$, or $\omega_1 = \lambda \leq \sigma \in \{\omega_\alpha : \alpha \leq I\}$ and $k = -2$.

Suppose $\{\gamma, \lambda, \sigma\} \subset \mathcal{H}_{\gamma,n}[\Theta]$ with $\forall \rho \geq \lambda[\Theta \subset \mathcal{H}_{\gamma,n}(\Psi_{\rho,n}\gamma)]$, and $\Gamma \in \Sigma^{\Sigma_{n+1}}(\lambda)$. Let $\mu = \sigma + 1$ if either σ is a regular cardinal or $\sigma = I$. Otherwise $\mu = \sigma$.

Moreover assume

$$(\mathcal{H}_{\gamma,n}, \Theta, \sigma, \text{ZFLK}_{k,n}) \vdash_\mu^a \Gamma$$

Then for $\hat{a} = \gamma + \omega^{\sigma(1+a)}$ we have

$$(\mathcal{H}_{\hat{a}+1,n}, \Theta, \lambda, \text{ZFLK}_{k,n}) \vdash_{\Psi_{\lambda,n}\hat{a}}^{\Psi_{\lambda,n}\hat{a}} \Gamma.$$

4.4 Lowering and eliminating higher Mahlo operations

In the section we eliminate inferences $(\mathcal{K} \in M_N)$ for Π_N^1 -indescribability.

In the following Lemma 4.20, let for the fixed n

$$(\mathcal{H}, \Theta, \text{ZFLK}_{k,n}) \vdash_b^a \Gamma :\Leftrightarrow (\mathcal{H}, \Theta, \mathcal{K}^+, \text{ZFLK}_{k,n}) \vdash_b^a \Gamma.$$

Recall that we have defined ordinals in (6) as follows: $b_n := \Psi_{\mathcal{K}^+,n}(\omega_{n-1}(I+1))$, $a_n := \varphi(b_n)(b_n)$, $\gamma_{k,n} := \omega_{n-1}(I+1) + 1 + a_n(N-k)$, and $\forall i < N-k = lh(\vec{\alpha}_{k,n})(\vec{\alpha}_{k,n}(i) = \gamma_{k+i,n})$.

Lemma 4.20 ($\text{FiX}^i(\text{ZFL})$)

Let $k > 0$. Suppose for an operator $\mathcal{H}_{\gamma,n}$ and an ordinal $a \leq a_n$

$$(\mathcal{H}_{\gamma,n}, \Theta, \text{ZFLK}_{k,n}) \vdash_{\mathcal{K}}^a \Lambda, \Gamma \tag{18}$$

where $\gamma \in \mathcal{H}_{\gamma,n}[\Theta]$ and $\gamma \leq \gamma_{k,n}$, $\Lambda \subset \Sigma^{\Sigma_n}(\lambda)$ for some regular cardinal $\lambda \in \mathcal{H}_{\gamma,n}[\Theta] \cap \mathcal{K}$, Γ consists of $\Pi_k^1(\mathcal{K})$ -sentences.

Let

$$\hat{a} := \gamma + a \leq \gamma_{k-1,n}$$

Then for any κ such that either $\kappa = \mathcal{K}$, or $\kappa \in Mh_{k-1,n}((\hat{a}) * \vec{\alpha}_{k,n})[\Theta \cup \{\kappa\}] \cap \mathcal{K}$ and $\mathcal{H}_{\hat{a},n}[\Theta](\kappa) \cap \mathcal{K} \subset \kappa$,

$$(\mathcal{H}_{\hat{a}}, \Theta \cup \{\kappa\}, \text{ZFLK}_{k-1,n}) \vdash_{\kappa}^{\kappa+\omega a} \Lambda, \Gamma^{(\kappa,\mathcal{K})} \tag{19}$$

holds.

Proof by induction on a . Let $\mathcal{H} = \mathcal{H}_{\gamma,n}$.

Let either $\kappa = \mathcal{K}$, or $\kappa \in Mh_{k-1,n}((\hat{a}) * \vec{\alpha}_{k,n})[\Theta \cup \{\kappa\}]$, $\kappa < \mathcal{K}$ and $\mathcal{H}_{\hat{a},n}[\Theta](\kappa) \cap \mathcal{K} \subset \kappa$. From $\mathcal{H}_{\hat{a},n}[\Theta](\kappa) \cap \mathcal{K} \subset \kappa$ and $\gamma \leq \hat{a}$ we see that

$$\lambda \in \mathcal{H}[\Theta](\kappa) \cap \mathcal{K} \subset \kappa \tag{20}$$

Also by (12) we have $k(\Gamma) \subset \mathcal{H}[\Theta] \cap (\mathcal{K} + 1)$ for the set Γ of $\Pi_k^1(\mathcal{K})$ -sentences. Hence $\Gamma^{(\kappa, \mathcal{K})}$ is a set of $\Pi_k^1(\kappa)$ -sentences.

By $\hat{a} \geq \gamma$, we obtain $\forall b \in K((\hat{a}) * \vec{\alpha})(b \leq \hat{a})$ if $\forall b \in K(\vec{\alpha})(b \leq \gamma)$, cf. (15).

Case 1. First consider the case when the last inference is a $(\mathcal{K} \in Mh_{k,n}(\vec{\alpha})[\Theta], \vec{\nu})$ where $\forall i < N - k(\vec{\alpha}(i) \leq \vec{\alpha}_{k,n})$. We have $a_\ell \in \mathcal{H}[\Theta] \cap a$, and $a_r(\rho) \in \mathcal{H}[\Theta \cup \{\rho\}] \cap a$. Δ is a finite set of $\Sigma_k^1(\mathcal{K})$ -sentences.

$$\frac{\{(\mathcal{H}, \Theta, \text{ZFLK}_{k,n}) \vdash_{\mathcal{K}}^{a_\ell} \Lambda, \Gamma, \neg \delta\}_{\delta \in \Delta} \quad \{(\mathcal{H}, \Theta \cup \{\rho\}, \text{ZFLK}_{k,n}) \vdash_{\mathcal{K}}^{a_r(\rho)} \Lambda, \Gamma, \Delta^{(\rho, \mathcal{K})}\}_{\rho \in H_k(\vec{\nu}, \gamma)}}{(\mathcal{H}, \Theta, \text{ZFLK}_{k,n}) \vdash_{\mathcal{K}}^a \Lambda, \Gamma}$$

where $H_k(\vec{\nu}, \gamma)$ is the resolvent class for $\mathcal{K} \in Mh_{k,n}(\vec{\alpha})[\Theta]$ with respect to a $\vec{\nu}$ and γ :

$$H_k(\vec{\nu}, \gamma) := \left\{ \rho \in \bigcap_{i < lh(\vec{\alpha})} Mh_{k+i,n}((\vec{\nu} \bullet \vec{\alpha})[i])[\Theta] \cap \mathcal{K} : \mathcal{H}_{\gamma,n}[\Theta](\rho) \cap \mathcal{K} \subset \rho \right\} \quad (16)$$

and $\vec{\nu}$ is a finite sequence of ordinals in $\mathcal{H}_{\vec{\nu},n}[\Theta](\mathcal{K}) \cap \vec{\alpha}$.

By (12) we have $k(\Delta) \subset \mathcal{H}[\Theta] \cap (\mathcal{K} + 1)$ for the set Δ of $\Sigma_k^1(\mathcal{K})$ -sentences. Hence for any $\rho \in H_k(\vec{\nu}, \gamma)$, $\Delta^{(\rho, \mathcal{K})}$ is a set of $\Sigma_k^1(\rho)$ -sentences, and a fortiori of $\Pi_k^1(\mathcal{K})$ -sentences with $(\Delta^{(\rho, \mathcal{K})})^{(\kappa, \mathcal{K})} = \Delta^{(\rho, \mathcal{K})}$.

Let $H_k(\vec{\nu}, \kappa, \hat{a})$ denote the class

$$\left\{ \rho \in \bigcap_{i < lh(\vec{\alpha})} Mh_{k+i,n}((\vec{\nu} \bullet \vec{\alpha})[i])[\Theta \cup \{\kappa\}] \cap \kappa : \mathcal{H}_{\hat{a},n}[\Theta \cup \{\kappa\}](\rho) \cap \kappa \subset \rho \right\}$$

By Proposition 2.5.3, $\mathcal{H}_{\hat{a},n}[\Theta \cup \{\kappa\}](\kappa) \cap \mathcal{K} \subset \kappa$ and $\hat{a} \geq \gamma$, we have

$$\rho \in H_k(\vec{\nu}, \kappa, \hat{a}) \Rightarrow \rho \in H_k(\vec{\nu}, \gamma) \quad (21)$$

Next we have $\widehat{a_r(\rho)} := \gamma + a_r(\rho) \in \mathcal{H}_{\widehat{a_r(\rho)},n}[\Theta \cup \{\rho\}](\mathcal{K}) \cap \hat{a}$ by $\widehat{a_r(\rho)} \geq \gamma$ and $a_r(\rho) < a$. If $\kappa < \mathcal{K}$, Propositions 2.5.1 and 2.5.5 with $a_r(\rho) < a$ we have $\kappa \in Mh_{k-1,n}((\widehat{a_r(\rho)}) * \vec{\alpha})[\Theta \cup \{\rho, \kappa\}]$, and $\mathcal{H}_{\widehat{a_r(\rho)},n}[\Theta \cup \{\rho\}](\kappa) \subset \mathcal{H}_{\hat{a},n}[\Theta](\kappa)$ for $\rho < \kappa$.

For each $\rho \in H_k(\vec{\nu}, \kappa, \hat{a})$, IH with (21) yields

$$(\mathcal{H}_{\hat{a}}, \Theta \cup \{\rho, \kappa\}, \text{ZFLK}_{k-1,n}) \vdash_{\kappa}^{\kappa + \omega a_r(\rho)} \Lambda, \Gamma^{(\kappa, \mathcal{K})}, \Delta^{(\rho, \mathcal{K})} \quad (22)$$

On the other hand we have $\widehat{a_\ell} := \gamma + a_\ell \in \mathcal{H}_{\widehat{a_\ell},n}[\Theta] \cap \hat{a}$. Let $\rho \in Mh_{k-1,n}((\widehat{a_\ell}) * \vec{\alpha})[\Theta \cup \{\kappa\}] \cap H_k(\vec{\nu}, \kappa, \hat{a})$. Then by Propositions 2.5.3 and 2.5.4 we have $\rho \in Mh_{k-1,n}((\widehat{a_\ell}) * \vec{\alpha})[\Theta \cup \{\rho\}] \cap \kappa$, and $\mathcal{H}_{\widehat{a_\ell},n}[\Theta](\rho) \cap \mathcal{K} \subset \rho$. Hence by IH we have for any $\rho \in Mh_{k-1,n}((\widehat{a_\ell}) * \vec{\alpha})[\Theta \cup \{\kappa\}] \cap H_k(\vec{\nu}, \kappa, \hat{a})$ and for any $\delta \in \Delta$,

$$(\mathcal{H}_{\hat{a}}, \Theta \cup \{\rho\}, \text{ZFLK}_{k-1,n}) \vdash_{\rho}^{\rho + \omega a_\ell} \Lambda, \Gamma^{(\rho, \mathcal{K})}, \neg \delta^{(\rho, \mathcal{K})} \quad (23)$$

Here note that $\rho > \lambda$, which is seen as in (20).

Now let

$$M_\ell := Mh_{k-1,n}((\widehat{a_\ell}) * \vec{\alpha})[\Theta \cup \{\kappa\}] \cap H_k(\vec{\nu}, \kappa, \hat{a}).$$

Then it is clear that M_ℓ is the resolvent class for $\kappa \in Mh_{k-1,n}((\hat{a}) * \vec{\alpha})[\Theta \cup \{\kappa\}]$ with respect to $\vec{\mu} = (\hat{a}_\ell) * \vec{\nu}$ and \hat{a} .

Since $\max\{\text{rk}_L(t) : t \in \mathbf{k}(\delta^{(\rho, \mathcal{K})})\} \leq \rho$, we have $\text{rk}(\delta^{(\rho, \mathcal{K})}) < \omega(\rho + 1) < \kappa$ by Proposition 4.10.4. From (22) and (23) by several (*cut*)'s of $\delta^{(\rho, \mathcal{K})}$ we obtain for $a(\rho) = \max\{a_\ell, a_r(\rho)\}$ and some $p < \omega$

$$(\mathcal{H}_{\hat{a}}, \Theta \cup \{\kappa, \rho\}, \text{ZFLK}_{k-1,n}) \vdash_{\kappa}^{\kappa + \omega a(\rho) + p} \Lambda, \Gamma^{(\kappa, \mathcal{K})}, \Gamma^{(\rho, \mathcal{K})}$$

for any $\rho \in M_\ell$.

By Inversion lemma 4.13 we obtain for any $B \in \mathcal{P}(\mathcal{K}) \cap L_{\mathcal{K}+}$

$$\{(\mathcal{H}_{\hat{a}}, \Theta \cup \{\kappa, \rho\}, \text{ZFLK}_{k-1,n}) \vdash_{\kappa}^{\kappa + \omega a(\rho) + p} \Lambda, \Gamma^{(\kappa, \mathcal{K})}, \Gamma(R_{B, \mathcal{K}})^{(\rho, \mathcal{K})} : \rho \in M_\ell\} \quad (24)$$

where $\Gamma(R_{B, \mathcal{K}}) = \{\tau(R_{B, \mathcal{K}}) : (\forall X \subset \mathcal{K} \tau(X)) \in \Gamma\}$ for $\Sigma_{k-1}^1(\mathcal{K})$ -formulae $\tau(X)$, and $\tau(R_{B, \mathcal{K}})^{(\rho, \mathcal{K})} \equiv \tau^{(\rho, \mathcal{K})}(R_{B, \rho})$.

On the other hand we have by Tautology lemma 4.14 for each $\tau(R_{B, \mathcal{K}})^{(\kappa, \mathcal{K})} \in \Gamma(R_{B, \mathcal{K}})^{(\kappa, \mathcal{K})}$

$$(\mathcal{H}, \Theta \cup \{\kappa\}, \text{ZFLK}_{k-1,n}) \vdash_0^{2\text{rk}(\tau(R_{B, \mathcal{K}})^{(\kappa, \mathcal{K})})} \Lambda, \Gamma(R_{B, \mathcal{K}})^{(\kappa, \mathcal{K})}, \neg \tau(R_{B, \mathcal{K}})^{(\kappa, \mathcal{K})} \quad (25)$$

where $2\text{rk}(\tau(R_{B, \mathcal{K}})^{(\kappa, \mathcal{K})}) \leq \kappa + p$ for some $p < \omega$ by Proposition 4.10.4. Note that $\text{rk}(\tau(R_{B, \mathcal{K}})^{(\kappa, \mathcal{K})})$ is independent from subsets B , where $\text{rk}(R_{B, \mathcal{K}}(c)^{(\kappa, \mathcal{K})}) = \text{rk}(R_{B, \kappa}(c)) = \kappa$, and $R_{B, \kappa}(c), \Gamma$ is an axiom if $c \in B \cap \kappa$.

Moreover we have $\sup\{2\text{rk}(\tau(R_{B, \mathcal{K}})^{(\kappa, \mathcal{K})}), \kappa + \omega a(\rho) + p : \rho \in M_\ell\} \leq \kappa + \omega a_0 + p \in \mathcal{H}[\Theta \cup \{\kappa\}]$ with $\sup\{a_\ell, a_r(\rho) : \rho \in H_k(\vec{\nu}, \gamma)\} \leq a_0 \in \mathcal{H}[\Theta] \cap a$ by (17).

Since $\neg \Gamma(R_{B, \mathcal{K}})^{(\kappa, \mathcal{K})}$ consists of $\Pi_{k-1}^1(\kappa)$ -sentences, by an inference rule $(\kappa \in Mh_{k-1,n}(\hat{a} * \vec{\alpha})[\Theta \cup \{\kappa\}], \vec{\mu})$ from (25) and (24) we conclude

$$(\mathcal{H}_{\hat{a}, n}, \Theta \cup \{\kappa\}, \text{ZFLK}_{k-1,n}) \vdash_{\kappa}^{\kappa + \omega a_0 + p + 1} \Lambda, \Gamma^{(\kappa, \mathcal{K})}, \Gamma(R_{B, \mathcal{K}})^{(\kappa, \mathcal{K})}$$

for any $B \in \mathcal{P}(\mathcal{K}) \cap L_{\mathcal{K}+}$, where $\hat{a} \leq \gamma_{k-1,n}$ and $\vec{\alpha}(i) \leq \gamma_{k+i,n} = \vec{\alpha}_{k,n}(i)$, $\kappa \in Mh_{k-1,n}(\hat{a} * \vec{\alpha})[\Theta \cup \{\kappa\}]$ by $\kappa \in Mh_{k-1,n}((\hat{a}) * \vec{\alpha}_{k,n})[\Theta \cup \{\kappa\}]$ when $\kappa < \mathcal{K}$.

Then by several $(\forall^2(\kappa))$'s we conclude

$$(\mathcal{H}_{\hat{a}, n}, \Theta \cup \{\kappa\}, \text{ZFLK}_{k-1,n}) \vdash_{\kappa}^{\kappa + \omega a} \Lambda, \Gamma^{(\kappa, \mathcal{K})}, \Gamma^{(\kappa, \mathcal{K})}$$

Case 2. Second consider the case when the last inference introduces a $\Pi_k^1(\mathcal{K})$ -sentence $(\forall X \subset \mathcal{K} \tau(X)) \in \Gamma$. For an $a_0 < a$ and $J = \mathcal{P}(\mathcal{K}) \cap L_{\mathcal{K}+}$

$$\frac{\{(\mathcal{H}, \Theta, \text{ZFLK}_{k,n}) \vdash_{\mathcal{K}}^{a_0} \Lambda, \Gamma, \tau(R_{B, \mathcal{K}}) : B \in J\}}{(\mathcal{H}, \Theta, \text{ZFLK}_{k,n}) \vdash_{\mathcal{K}}^a \Lambda, \Gamma} (\forall^2(\mathcal{K}))$$

IH yields for each $B \in J$ and $(R_{B, \mathcal{K}})^{(\kappa, \mathcal{K})} = R_{B, \kappa}$

$$(\mathcal{H}_{\hat{a}}, \Theta \cup \{\kappa\}, \text{ZFLK}_{k-1,n}) \vdash_{\kappa}^{\kappa + \omega a_0} \Lambda, \Gamma^{(\kappa, \mathcal{K})}, \tau(R_{B, \mathcal{K}})^{(\kappa, \mathcal{K})}$$

$(\forall^2(\kappa))$ yields (19) with $(\forall X \subset \kappa \tau(X))^{(\kappa, \mathcal{K})} \in \Gamma^{(\kappa, \mathcal{K})}$.

Case 3. Third consider the case: for a true literal $M \equiv (R_{B,\sigma}(d))$, $M \in \Lambda \cup \Gamma$, where $\sigma \leq \mathcal{K}$ and $d \in B \cap \sigma$.

$$\frac{}{(\mathcal{H}, \Theta, \text{ZFLK}_{k,n}) \vdash_{\mathcal{K}}^a \Lambda, \Gamma} (\wedge)$$

First consider the case when $\sigma < \mathcal{K}$. Then $M^{(\kappa, \mathcal{K})} \equiv M \equiv (R_{B,\sigma}(d)) \in \Lambda \cup \Gamma^{(\kappa, \mathcal{K})}$.

Second consider the case when $\sigma = \mathcal{K}$. Then $M^{(\kappa, \mathcal{K})} \equiv (R_{B,\kappa}(d)) \in \Gamma^{(\kappa, \mathcal{K})}$. It suffices to show $d = \text{rk}_L(d) < \kappa$. We have $d \in \mathbf{k}(R_{B,\mathcal{K}}(d)) \cap \mathcal{K} \subset \mathcal{H}[\Theta] \cap \mathcal{K} \subset \mathcal{K}$ by (12) and (20).

Case 4. Fourth consider the case when the last inference introduces a $\Pi_0^1(\mathcal{K})$ -sentence $(\exists x < \mathcal{K} \tau(x)) \in \Gamma$. For a $d < \mathcal{K}$

$$\frac{(\mathcal{H}, \Theta, \text{ZFLK}_{k,n}) \vdash_{\mathcal{K}}^{a_0} \Lambda, \Gamma, \tau(d)}{(\mathcal{H}, \Theta, \text{ZFLK}_{k,n}) \vdash_{\mathcal{K}}^a \Lambda, \Gamma} (\vee)$$

where if $(\exists x < \mathcal{K} \tau(x)) \in \Sigma_n$ (this means $(\exists x < \mathcal{K} \tau(x)) \in \Delta_0$), then $d = (\mu x < \mathcal{K} \tau(x)) \in \mathcal{H}[\Theta]$, cf. (1) in Definition 10.

Without loss of generality we can assume that $d \in \mathbf{k}(\tau(d)) \subset \mathcal{H}[\Theta]$. Then as in **Case 3** we see that $d < \kappa < \kappa + \omega a$, cf. (13). Moreover when $(\exists x < \mathcal{K} \tau(x)) \in \Sigma_n$, we have $(\exists x < \mathcal{K} \tau(x))^{(\kappa, \mathcal{K})} \in \Sigma_n$ and $d = (\mu x < \mathcal{K} \tau(x)) = (\mu x < \kappa \tau^{(\kappa, \mathcal{K})}(x))$ since L_κ is an elementary submodel of $L_{\mathcal{K}}$, which is seen from $\mathcal{H}_{\hat{a},n}[\Theta](\kappa) \cap \mathcal{K} \subset \kappa$.

IH yields with $(\exists x < \mathcal{K} \tau(x))^{(\kappa, \mathcal{K})} \equiv \exists x < \kappa \tau(x)^{(\kappa, \mathcal{K})} \in \Gamma^{(\kappa, \mathcal{K})}$

$$\frac{(\mathcal{H}_{\hat{a}}, \Theta \cup \{\kappa\}, \text{ZFLK}_{k-1,n}) \vdash_{\kappa}^{\kappa + \omega a_0} \Lambda, \Gamma^{(\kappa, \mathcal{K})}, \tau(d)^{(\kappa, \mathcal{K})}}{(\mathcal{H}_{\hat{a}}, \Theta \cup \{\kappa\}, \text{ZFLK}_{k-1,n}) \vdash_{\kappa}^{\kappa + \omega a} \Lambda, \Gamma^{(\kappa, \mathcal{K})}} (\vee)$$

Case 5. Fifth consider the case when the last inference introduces a $\Pi_0^1(\mathcal{K})$ -sentence $(\forall x < \mathcal{K} \tau(x)) \in \Gamma$.

$$\frac{\{(\mathcal{H}, \Theta \cup \{\alpha\}, \text{ZFLK}_{k,n}) \vdash_{\mathcal{K}}^{a(\alpha)} \Lambda, \Gamma, \tau(\alpha) : \alpha \in J\}}{(\mathcal{H}, \Theta, \text{ZFLK}_{k,n}) \vdash_{\mathcal{K}}^a \Lambda, \Gamma} (\wedge)$$

where if $(\forall x < \mathcal{K} \tau(x)) \in \Sigma_n$, then $J = \{d\}$ for $d = (\mu x < \mathcal{K} \neg \tau(x))$, and $J = \mathcal{K}$ otherwise. In the former case we see (19) from IH and $(\mu x < \mathcal{K} \neg \tau(x)) = (\mu x < \kappa \neg \tau^{(\kappa, \mathcal{K})}(x))$.

In what follows suppose $(\forall x < \mathcal{K} \tau(x)) \notin \Sigma_n$. Then $(\forall x < \kappa \tau^{(\kappa, \mathcal{K})}(x)) \notin \Sigma_n$. Let $\alpha < \kappa$. We have $\mathcal{H}_{\widehat{a(\alpha)},n}[\Theta \cup \{\alpha\}](\kappa) = \mathcal{H}_{\widehat{a(\alpha)},n}[\Theta](\kappa)$, and $\kappa \in Mh_{k-1,n}((\widehat{a(\alpha)}) * \vec{\alpha}_{k,n}[\Theta \cup \{\alpha\}])$ by $\kappa \in Mh_{k-1,n}((\hat{a}) * \vec{\alpha}_{k,n}[\Theta])$ and Proposition 2.5.5 when $\kappa < \mathcal{K}$.

IH yields for $\alpha < \kappa$

$$(\mathcal{H}_{\hat{a}}, \Theta \cup \{\kappa, \alpha\}, \text{ZFLK}_{k-1,n}) \vdash_{\kappa}^{\kappa + \omega a(\alpha)} \Lambda, \Gamma^{(\kappa, \mathcal{K})}, \tau(\alpha)^{(\kappa, \mathcal{K})}$$

(\bigwedge) yields (19) with $\forall x < \kappa \tau(x)^{(\kappa, \mathcal{K})} \equiv (\forall x < \mathcal{K} \tau(x))^{(\kappa, \mathcal{K})} \in \Gamma^{(\kappa, \mathcal{K})}$.

Case 6. Sixth consider the case when the last inference introduces a $\Sigma_{k-1}^1(\mathcal{K})$ -sentence $(\exists X \subset \mathcal{K} \tau(X)) \in \Gamma$. For a $B \in \mathcal{P}(\mathcal{K}) \cap L_{\mathcal{K}+}$

$$\frac{(\mathcal{H}, \Theta, \text{ZFLK}_{k,n}) \vdash_{\mathcal{K}}^{a_0} \Lambda, \Gamma, \tau(R_{B,\mathcal{K}})}{(\mathcal{H}, \Theta, \text{ZFLK}_{k,n}) \vdash_{\mathcal{K}}^a \Lambda, \Gamma} (\exists^2(\mathcal{K}))$$

IH with $(R_{B,\mathcal{K}})^{(\kappa, \mathcal{K})} \equiv R_{B,\kappa}$ yields

$$\frac{(\mathcal{H}_{\hat{a}}, \Theta \cup \{\kappa\}, \text{ZFLK}_{k-1,n}) \vdash_{\kappa}^{\kappa+\omega a_0} \Lambda, \Gamma^{(\kappa, \mathcal{K})}, \tau^{(\kappa, \mathcal{K})}(R_{B,\kappa})}{(\mathcal{H}_{\hat{a}}, \Theta \cup \{\kappa\}, \text{ZFLK}_{k-1,n}) \vdash_{\kappa}^{\kappa+\omega a} \Lambda, \Gamma^{(\kappa, \mathcal{K})}} (\exists^2(\kappa))$$

Case 7. Seventh consider the case when the last inference is a $(\sigma \in Mh_{j,n}(\vec{\alpha})[\Theta], \vec{\nu})$ for a $\sigma < \mathcal{K}$ and a $j < N$:

$$\frac{\{(\mathcal{H}, \Theta, \text{ZFLK}_{k,n}) \vdash_{\mathcal{K}}^{a_\ell} \Lambda, \Gamma, \neg \delta\}_{\delta \in \Delta} \quad \{(\mathcal{H}, \Theta \cup \{\rho\}, \text{ZFLK}_{k,n}) \vdash_{\mathcal{K}}^{a_r(\rho)} \Lambda, \Gamma, \Delta^{(\rho, \sigma)}\}_{\rho \in H_j(\vec{\nu}, \gamma)}}{(\mathcal{H}, \Theta, \text{ZFLK}_{k,n}) \vdash_{\mathcal{K}}^a \Lambda, \Gamma}$$

where Δ is a finite set of $\Sigma_j^1(\sigma)$ -sentences, and $H_j(\vec{\nu}, \gamma) = \{\rho \in \bigcap_{i < lh(\vec{\alpha})} Mh_{j+i,n}((\vec{\nu} \bullet \vec{\alpha})[i][\Theta] \cap \sigma : \mathcal{H}_{\gamma,n}[\Theta](\rho) \cap \sigma \subset \rho)\}$ is the resolvent class for $\sigma \in Mh_{j,n}(\vec{\alpha})[\Theta]$ with respect to a $\vec{\nu}$ and γ .

We have $\sigma \in \mathcal{H}[\Theta] \cap \mathcal{K} \subset \kappa$ by (14) and (20). Hence $\sigma < \kappa$, and $\Delta \subset \Sigma_0^2(\sigma) \subset \Delta_0(\kappa)$ and $\delta^{(\kappa, \mathcal{K})} \equiv \delta$ for any $\delta \in \Delta$.

Let $H_j(\vec{\nu}, \kappa, \hat{a}) = \{\rho \in \bigcap_{i < lh(\vec{\alpha})} Mh_{j+i,n}((\vec{\nu} \bullet \vec{\alpha})[i][\Theta] \cup \{\kappa\}) \cap \sigma : \mathcal{H}_{\hat{a},n}[\Theta \cup \{\kappa\}](\rho) \cap \sigma \subset \rho\}$ be a resolvent class for $\sigma \in Mh_{j,n}(\vec{\alpha})[\Theta \cup \{\kappa\}]$. Then $H_j(\vec{\nu}, \kappa, \hat{a}) \subset H_j(\vec{\nu}, \gamma)$ as in **Case 1**.

From IH we obtain the assertion (19) by an inference rule $(\sigma \in Mh_{j,n}(\vec{\alpha})[\Theta \cup \{\kappa\}], \vec{\nu})$ with the resolvent class $H_j(\vec{\nu}, \kappa, \hat{a})$.

Case 8. Eighth consider the case when the last inference is a (cut) .

$$\frac{(\mathcal{H}, \Theta, \text{ZFLK}_{k,n}) \vdash_{\mathcal{K}}^{a_0} \Lambda, \Gamma, \neg C \quad (\mathcal{H}, \Theta, \text{ZFLK}_{k,n}) \vdash_{\mathcal{K}}^{a_0} C, \Lambda, \Gamma}{(\mathcal{H}, \Theta, \text{ZFLK}_{k,n}) \vdash_{\mathcal{K}}^a \Lambda, \Gamma} (cut)$$

where $a_0 < a$ and $\text{rk}(C) < \mathcal{K}$. Then $C \in \Delta_0(\mathcal{K})$ by Proposition 4.10.6. On the other side we have $\mathbf{k}(C) \subset \mathcal{K}$ by Proposition 4.10.4. Then $\mathbf{k}(C) \subset \mathcal{H}[\Theta] \cap \mathcal{K} \subset \kappa$ by (12) and (20). Hence $C^{(\kappa, \mathcal{K})} \equiv C$ and $\text{rk}(C^{(\kappa, \mathcal{K})}) < \kappa$ again by Proposition 4.10.4.

IH yields

$$(\mathcal{H}_{\hat{a}}, \Theta \cup \{\kappa\}, \text{ZFLK}_{k-1,n}) \vdash_{\kappa}^{\kappa+\omega a_0} \Lambda, \Gamma^{(\kappa, \mathcal{K})}, \neg C^{(\kappa, \mathcal{K})}$$

and

$$(\mathcal{H}_{\hat{a}}, \Theta \cup \{\kappa\}, \text{ZFLK}_{k-1,n}) \vdash_{\kappa}^{\kappa+\omega a_0} C^{(\kappa, \mathcal{K})}, \Lambda, \Gamma^{(\kappa, \mathcal{K})}$$

Then by a (cut) we obtain

$$(\mathcal{H}_{\hat{a}}, \Theta \cup \{\kappa\}, \text{ZFLK}_{k-1,n}) \vdash_{\kappa}^{\kappa+\omega a} \Lambda, \Gamma^{(\kappa, \mathcal{K})}$$

Case 9. Ninth consider the case when the last inference is an (\mathbf{F}) where either $F = F_{x \cup \{\lambda\}}^{\Sigma_1}$ for a $\lambda \in \text{Reg}$ or $F = F_x^{\Sigma_n}$. Let $F''' A_0 \equiv A \in \text{rng}(F)$. Then $A_0 \in \Sigma_n$. IH yields (19).

All other cases are seen easily from IH. \square

Corollary 4.21 ($\text{FiX}^i(\text{ZFL})$)

Let $k > 0$. Suppose for the operator $\mathcal{H}_{\gamma_k, n}$

$$(\mathcal{H}_{\gamma_k, n}, \emptyset, \text{ZFLK}_{k, n}) \vdash_{\mathcal{K}}^{a_n} \Lambda, \Gamma$$

where $\Lambda \subset \Sigma^{\Sigma_n}(\omega_1)$, Γ consists of $\Pi_k^1(\mathcal{K})$ -sentences. Then the following holds

$$(\mathcal{H}_{\gamma_{k-1}, n}, \emptyset, \text{ZFLK}_{k-1, n}) \vdash_{\mathcal{K}}^{a_n} \Lambda, \Gamma$$

Proof. By Lemma 4.20 with $\gamma_{k, n} \in \mathcal{H}_{\gamma_k, n}$ and $\kappa = \mathcal{K}$ we have

$$(\mathcal{H}_{\gamma_{k, n} + a_n, n}, \{\mathcal{K}\}, \text{ZFLK}_{k-1, n}) \vdash_{\mathcal{K}}^{\mathcal{K} + \omega a_n} \Lambda, \Gamma$$

in other words

$$(\mathcal{H}_{\gamma_{k, n} + a_n, n}, \emptyset, \text{ZFLK}_{k-1, n}) \vdash_{\mathcal{K}}^{\mathcal{K} + \omega a_n} \Lambda, \Gamma$$

Note that $\gamma_{k, n} + a_n = \gamma_{k-1, n}$ and $\mathcal{K} + \omega a_n = a_n$. \square

Corollary 4.22 ($\text{FiX}^i(\text{ZFL})$)

Let $0 \leq k < N$. Suppose for the operator $\mathcal{H}_{\gamma_N, n}$

$$(\mathcal{H}_{\gamma_N, n}, \emptyset, \mathcal{K}^+, \text{ZFLK}_{N, n}) \vdash_{\mathcal{K}}^{a_n} \Lambda, \Gamma$$

where $\Lambda \subset \Sigma^{\Sigma_n}(\omega_1)$, Γ consists of $\Pi_{k+1}^1(\mathcal{K})$ -sentences.

Then the following holds

$$(\mathcal{H}_{\gamma_k, n}, \emptyset, \mathcal{K}^+, \text{ZFLK}_{k, n}) \vdash_{\mathcal{K}}^{a_n} \Lambda, \Gamma$$

Proof. This is seen from Corollary 4.21. \square

5 Theorems

Let us conclude two theorems.

Lemma 5.1 (Embedding of Axioms)

For each axiom A in $\text{T}_N(\mathcal{K}, I, n)$, there is an $m < \omega$ such that for any operator $\mathcal{H} = \mathcal{H}_{\gamma, n}$, the fact that $(\mathcal{H}, \emptyset, I, \text{ZFLK}_{N, n}) \vdash_{I+m}^{I+2} A$ is provable in $\text{FiX}^i(\text{ZFL})$.

Proof. In this proof let us write (\mathcal{H}, Θ) for $(\mathcal{H}, \Theta, I, \text{ZFLK}_{N, n})$.

Let us consider the axiom for N -indescribability of \mathcal{K}

$$\forall x[x \subset \mathcal{K} \rightarrow \varphi_0(x) \rightarrow \exists \rho < \mathcal{K}(\varphi_0^{(\rho, \mathcal{K})}(x \cap \rho))] \quad (5)$$

where φ_0 is the $\Pi_N^1(\mathcal{K})$ -translation of a Π_N^1 -formula φ . Let φ_1 be the $\Pi_N^1(\mathcal{K})$ -sentence obtained from φ . Suppose that φ contains a second-order quantifier.

First let $B \subset \mathcal{K}$. Then by Tautology lemma 4.14 with $k(\varphi_1^{(\rho, \mathcal{K})}(R_{B, \rho})) = \{\rho\}$

$$(\mathcal{H}, \emptyset) \vdash_0^{2\text{rk}(\varphi_1(R_{B, \mathcal{K}}))} \neg \varphi_1(R_{B, \mathcal{K}}), \varphi_1(R_{B, \mathcal{K}})$$

and for each $\rho < \mathcal{K}$

$$(\mathcal{H}, \{\rho\}) \vdash_0^{2\text{rk}(\varphi_1^{(\rho, \mathcal{K})}(R_{B, \rho})) + 1} \neg \varphi_1^{(\rho, \mathcal{K})}(R_{B, \rho}), \exists \rho < \mathcal{K} \varphi_1^{(\rho, \mathcal{K})}(R_{B, \rho})$$

By Proposition 4.10.4 we have $\rho \leq \text{rk}(\varphi_1^{(\rho, \mathcal{K})}(R_{B, \rho})) < \omega\rho + \omega$, and hence $\text{rk}(\varphi_1^{(\rho, \mathcal{K})}(R_{B, \rho})) < \text{rk}(\varphi_1(R_{B, \mathcal{K}})) < \mathcal{K} + \omega$ for any $\rho < \mathcal{K}$. By the inference rule ($\mathcal{K} \in M_N$) we obtain

$$(\mathcal{H}, \emptyset) \vdash_{\mathcal{K} + \omega}^{\mathcal{K} + \omega} \neg \varphi_1(R_{B, \mathcal{K}}), \exists \rho < \mathcal{K} \varphi_1^{(\rho, \mathcal{K})}(R_{B, \rho})$$

On the other hand we have by Proposition 4.15 for $d = d(\varphi) < \omega$

$$(\mathcal{H}, \{B\}) \vdash_0^{2d} \neg \varphi_0(B), \varphi_1(R_{B, \mathcal{K}})$$

and

$$(\mathcal{H}, \{B\}) \vdash_0^{2d+2} \neg \exists \rho < \mathcal{K} \varphi_1^{(\rho, \mathcal{K})}(R_{B, \rho}), \exists \rho < \mathcal{K} \varphi_0^{(\rho, \mathcal{K})}(B \cap \rho)$$

Two (*cut*)'s yield

$$(\mathcal{H}, \{B\}) \vdash_{\mathcal{K} + \omega}^{\mathcal{K} + \omega + 2} \neg \varphi_0(B), \exists \rho < \mathcal{K} \varphi_0^{(\rho, \mathcal{K})}(B \cap \rho)$$

and

$$(\mathcal{H}, \{B\}) \vdash_{\mathcal{K} + \omega}^{\mathcal{K} + \omega + 5} B \not\subset \mathcal{K} \vee (\neg \varphi_0(B) \vee \exists \rho < \mathcal{K} \varphi_0^{(\rho, \mathcal{K})}(B \cap \rho))$$

Second for $a \not\subset \mathcal{K}$ we have $(\mathcal{H}, \{a\}) \vdash_0^1 a \not\subset \mathcal{K}$ and

$$(\mathcal{H}, \{a\}) \vdash_0^2 a \not\subset \mathcal{K} \vee (\neg \varphi_0(a) \vee \exists \rho < \mathcal{K} \varphi_0^{(\rho, \mathcal{K})}(a \cap \rho))$$

Therefore an inference rule (\wedge) yields

$$(\mathcal{H}, \emptyset) \vdash_{\mathcal{K} + \omega}^{\mathcal{K} + \omega + 6} \forall x[x \subset \mathcal{K} \rightarrow \varphi_0(x) \rightarrow \exists \rho < \mathcal{K}(\varphi_0^{(\rho, \mathcal{K})}(x \cap \rho))]$$

Other axioms are seen as in [4]. □

Lemma 5.2 (Embedding)

1. If $\text{T}_N(\mathcal{K}, I, n) \vdash \Gamma$ for sets Γ of sentences, there are $m, k < \omega$ such that for any operator $\mathcal{H} = \mathcal{H}_{\gamma, n}$, the fact that $(\mathcal{H}, \emptyset, I, \text{ZFLK}_{N, n}) \vdash_{I+m}^{I \cdot 2 + k} \Gamma$ is provable in $\text{FiX}^i(\text{ZFL})$.
2. If $\text{ZFL} \vdash \Gamma$ for sets Γ of sentences, there are $n, m, k < \omega$ such that for any operator $\mathcal{H} = \mathcal{H}_{\gamma, n}$, the fact that $(\mathcal{H}, \emptyset, I, \text{ZFLK}_{-2, n}) \vdash_{I+m}^{I \cdot 2 + k} \Gamma$ is provable in $\text{FiX}^i(\text{ZFL})$.

Theorem 5.3 Suppose for a $\Sigma_{k+2}^1(\mathcal{K})$ -sentence φ ,

$$\text{ZFLK}_N \vdash \varphi$$

Then we can find an $n < \omega$ such that

$$\text{ZFLK}_{k,n} \vdash \varphi$$

In short, ZFLK_N is $\Sigma_{k+2}^1(\mathcal{K})$ -conservative over $\text{ZFLK}_k = \bigcup_{n < \omega} \text{ZFLK}_{k,n} = \text{ZFL} + \{\mathcal{K} \in \text{Mh}_{k,n}(\vec{\alpha}_{k,n}) : n < \omega\}$.

Proof. Suppose a $\Sigma_{k+2}^1(\mathcal{K})$ -sentence $\varphi \equiv \exists X \subset \mathcal{K} \theta(X)$ is provable in ZFLK_N . Let $B := \mu B \in \mathcal{P}(\mathcal{K}) \cap L_{\mathcal{K}^+}(\theta(R_{B,\mathcal{K}}))$. By Embedding lemma 5.2 there exist $m, p < \omega$ such that $(\mathcal{H}_{0,n}, \emptyset, I, \text{ZFLK}_{N,n}) \vdash_{I+m}^{I \cdot 2+p} \theta(R_{B,\mathcal{K}})$ with $n = m+3$. Let us work temporarily in $\text{FiX}^i(\text{ZFL})$. By Predicative elimination 4.18.1 and 4.18.3, $(\mathcal{H}_{\omega_{m-1}(I \cdot 2+p),n}, \emptyset, I, \text{ZFLK}_{N,n}) \vdash_I^{\omega_m(I \cdot 2+p)} \theta(R_{B,\mathcal{K}})$. By Collapsing lemma 4.19, $(\mathcal{H}_{\gamma_{N,n},n}, \emptyset, \mathcal{K}^+, \text{ZFLK}_{N,n}) \vdash_{b_n}^{b_n} \theta(R_{B,\mathcal{K}})$ for $b_n = \Psi_{\mathcal{K}^+,n}(c)$ for $c = \omega_{m+2}(I+1) > \omega_{m+1}(I \cdot 2 + p)$. By Predicative elimination 4.18.1 for $a_n = \varphi b_n b_n$,

$$(\mathcal{H}_{\gamma_{N,n},n}, \emptyset, \mathcal{K}^+, \text{ZFLK}_{N,n}) \vdash_{\mathcal{K}}^{a_n} \theta(R_{B,\mathcal{K}})$$

By Corollary 4.22

$$(\mathcal{H}_{\gamma_{k,n},n}, \emptyset, \mathcal{K}^+, \text{ZFLK}_{k,n}) \vdash_{\mathcal{K}}^{a_n} \theta(R_{B,\mathcal{K}})$$

Note that any inference rule $(\pi \in \text{Mh}_{i,n}(\vec{\alpha})[\Theta], \vec{\nu})$ is correct for $\pi < \mathcal{K}$ provably in ZFL . Also inference rules $(\mathcal{K} \in \text{Mh}_{k,n}(\vec{\alpha})[\Theta], \vec{\nu})$ occur only for $\Theta \subset (\mathcal{K}+1)$ in the derivation establishing the fact $(\mathcal{H}_{\gamma_{k,n},n}, \emptyset, \mathcal{K}^+, \text{ZFLK}_{k,n}) \vdash_{\mathcal{K}}^{a_n} \theta(R_{B,\mathcal{K}})$. Hence $\mathcal{K} \in \text{Mh}_{k,n}(\vec{\alpha})[\Theta]$ is equivalent to $\mathcal{K} \in \text{Mh}_{k,n}(\vec{\alpha})[\emptyset]$ by Proposition 2.5.5. Moreover the rule $(\mathcal{K} \in \text{Mh}_{k,n}(\vec{\alpha}), \vec{\nu})$ with $\vec{\alpha}(i) \leq \vec{\alpha}_{k,n}(i)$ is correct assuming $\mathcal{K} \in \text{Mh}_{k,n}(\vec{\alpha}_{k,n})$, we conclude by induction on $a_n < \mathcal{K}^+$ that,

$$\text{FiX}^i(\text{ZFLK}_{k,n}) \vdash \theta(B)$$

and hence by Theorem 4.2

$$\text{ZFLK}_{k,n} \vdash \varphi$$

□

Theorem 5.4 Suppose for a first-order formula φ

$$\text{ZFLK}_N \vdash \exists x \in L_{\omega_1} \varphi(x)$$

Then we can find an $n < \omega$ such that

$$\text{ZFL} \vdash \exists \kappa < \mathcal{K} (\kappa = \Psi_{\mathcal{K},n}^{\vec{\alpha}_{0,n},\emptyset}(\omega_{n-1}(I+1))) \rightarrow \forall \alpha [\alpha = \Psi_{\omega_1,n}(\omega_{n-1}(I+1)) \rightarrow \exists x \in L_\alpha \varphi(x)]$$

Proof. As in the proof of Theorem 5.3, we see for $n_0 = m_0 + 3$ and $(\exists x \in L_{\omega_1} \varphi(x)) \in \Sigma^{\Sigma_{n_0}}(\omega_1)$

$$(\mathcal{H}_{\gamma_1, n_0, n_0}, \emptyset, \mathcal{K}^+, \text{ZFLK}_{1, n_0}) \vdash_{\mathcal{K}}^{a_{n_0}} \exists x \in L_{\omega_1} \varphi(x)$$

Let $\kappa = \Psi_{\mathcal{K}, n_0}^{\vec{\alpha}_{0, n_0}, \emptyset}(\gamma_{0, n_0}) < \mathcal{K}$. Then Lemma 4.20 yields for $\gamma_{0, n_0} = \gamma_{1, n_0} + a_{n_0}$, $\kappa + \omega a_{n_0} = a_{n_0}$ and $\vec{\alpha}_{0, n_0} = (\gamma_{0, n_0}) * \vec{\alpha}_{1, n_0}$ that

$$(\mathcal{H}_{\gamma_{0, n_0}, n_0}, \{\kappa\}, \mathcal{K}^+, \text{ZFLK}_{0, n_0}) \vdash_{\kappa}^{a_{n_0}} \exists x \in L_{\omega_1} \varphi(x)$$

Note that there occurs no inference rule $(\mathcal{K} \in Mh_{i, n_0}(\vec{\alpha})[\Theta], \vec{\nu})$ for any $k, \vec{\alpha}$ in the derivation establishing this fact. In other words we have

$$(\mathcal{H}_{\gamma_{0, n_0}, n_0}, \{\kappa\}, \mathcal{K}^+, \text{ZFLK}_{-1, n_0}) \vdash_{\kappa}^{a_{n_0}} \exists x \in L_{\omega_1} \varphi(x)$$

We see by induction up to $a_{n_0} < \mathcal{K}^+$ that

$$\text{FiX}^i(\text{ZFL}) \vdash \exists \kappa < \mathcal{K}(\kappa = \Psi_{\mathcal{K}, n_0}^{\vec{\alpha}_{0, n_0}, \emptyset}(\gamma_{0, n_0})) \rightarrow \exists x \in L_{\omega_1} \varphi(x)$$

Hence by Theorem 4.2

$$\text{ZFL} \vdash \exists \kappa < \mathcal{K}(\kappa = \Psi_{\mathcal{K}, n_0}^{\vec{\alpha}_{0, n_0}, \emptyset}(\gamma_{0, n_0})) \rightarrow \exists x \in L_{\omega_1} \varphi(x)$$

Again by Embedding lemma 5.2, Predicative elimination 4.18.1 and 4.18.3, Collapsing lemma 4.19, we can find m and $n > n_0$ such that $\exists \kappa(\kappa = \Psi_{\mathcal{K}, n_0}^{\vec{\alpha}_{0, n_0}, \emptyset}(\gamma_{0, n_0})) \in \Pi_n$ and

$$(\mathcal{H}_{b+1, n}, \emptyset, \omega_1, \text{ZFLK}_{-2, n}) \vdash_{\beta}^{\beta} \exists \kappa(\kappa = \Psi_{\mathcal{K}, n_0}^{\vec{\alpha}_{0, n_0}, \emptyset}(\gamma_{0, n_0})) \rightarrow \exists x \in L_{\omega_1} \varphi(x)$$

for $\beta = \Psi_{\omega_1, n}(b)$, $b = \omega_{m+1}(I \cdot 2 + \omega)$. Boundedness lemma 4.17 yields

$$(\mathcal{H}_{b+1, n}, \emptyset, \omega_1, \text{ZFLK}_{-2, n}) \vdash_{\beta}^{\beta} \exists \kappa(\kappa = \Psi_{\mathcal{K}, n_0}^{\vec{\alpha}_{0, n_0}, \emptyset}(\gamma_{0, n_0})) \rightarrow \exists x \in L_{\beta} \varphi(x)$$

Thus

$$\text{FiX}^i(\text{ZFL}) \vdash \exists \kappa(\kappa = \Psi_{\mathcal{K}, n_0}^{\vec{\alpha}_{0, n_0}, \emptyset}(\gamma_{0, n_0})) \rightarrow \forall \beta[\beta = \Psi_{\omega_1, n}(\omega_{m+1}(I \cdot 2 + \omega)) \rightarrow \exists x \in L_{\beta} \varphi(x)]$$

and hence by Theorem 4.2

$$\text{ZFL} \vdash \exists \kappa(\kappa = \Psi_{\mathcal{K}, n_0}^{\vec{\alpha}_{0, n_0}, \emptyset}(\gamma_{0, n_0})) \rightarrow \forall \beta[\beta = \Psi_{\omega_1, n}(\omega_{m+1}(I \cdot 2 + \omega)) \rightarrow \exists x \in L_{\beta} \varphi(x)]$$

Finally we have for $n > n_0, m + 2$

$$\text{ZFL} \vdash \Psi_{\omega_1, n}(\omega_{n-1}(I + 1)) > \Psi_{\omega_1, n}(\omega_{m+1}(I \cdot 2 + \omega))$$

and

$$\text{ZFL} \vdash \exists \kappa < \mathcal{K}(\kappa = \Psi_{\mathcal{K}, n}^{\vec{\alpha}_{0, n}, \emptyset}(\gamma_{0, n})) \rightarrow \exists \kappa < \mathcal{K}(\kappa = \Psi_{\mathcal{K}, n_0}^{\vec{\alpha}_{0, n_0}, \emptyset}(\gamma_{0, n_0})) \quad (26)$$

Consider the latter (26). Let $n > n_0$. Then $\mathcal{H}_{\gamma,n}[\Theta] \supset \mathcal{H}_{\gamma,n_0}[\Theta]$ for any γ, Θ . Hence $b_n = \Psi_{\mathcal{K}^+,n}(\omega_{n-1}(I+1)) \geq \Psi_{\mathcal{K}^+,n_0}(\omega_{n_0-1}(I+1)) = b_{n_0}$, $\gamma_{0,n} > \gamma_{0,n_0}$, and $\vec{\alpha}_{0,n} > \vec{\alpha}_{0,n_0}$. Let $\kappa = \Psi_{\mathcal{K},n}^{\vec{\alpha}_{0,n},\emptyset}(\gamma_{0,n}) < \mathcal{K}$. Then $\kappa \in \bigcap_{i < N} Mh_{i,n}(\vec{\alpha}_{i,n})[\{\kappa\}]$ and $\mathcal{H}_{\gamma_{0,n},n}(\kappa) \cap \mathcal{K} \subset \kappa$. Hence $\mathcal{H}_{\gamma_{0,n_0},n_0}(\kappa) \cap \mathcal{K} \subset \kappa$. In general we see by induction on κ using the definition (2) that

$$\kappa \in Mh_{i,n}(\vec{\alpha})[\Theta] \Rightarrow \kappa \in Mh_{i,n_0}(\vec{\alpha})[\Theta]$$

Therefore we obtain $\kappa \in \bigcap_{i < N} Mh_{i,n_0}(\vec{\alpha}_{i,n_0})[\{\kappa\}]$ by Proposition 2.5.1 with $\vec{\alpha}_{0,n} > \vec{\alpha}_{0,n_0}$, and $\Psi_{\mathcal{K},n_0}^{\vec{\alpha}_{0,n_0},\emptyset}(\gamma_{0,n_0}) \leq \Psi_{\mathcal{K},n}^{\vec{\alpha}_{0,n},\emptyset}(\gamma_{0,n}) = \kappa < \mathcal{K}$ by the definition (4).

Consequently we obtain

$$\text{ZFL} \vdash \exists \kappa < \mathcal{K} (\kappa = \Psi_{\mathcal{K},n}^{\vec{\alpha}_{0,n},\emptyset}(\omega_{n-1}(I+1))) \rightarrow \forall \alpha [\alpha = \Psi_{\omega_1,n}(\omega_{n-1}(I+1)) \rightarrow \exists x \in L_\alpha \varphi(x)]$$

□

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